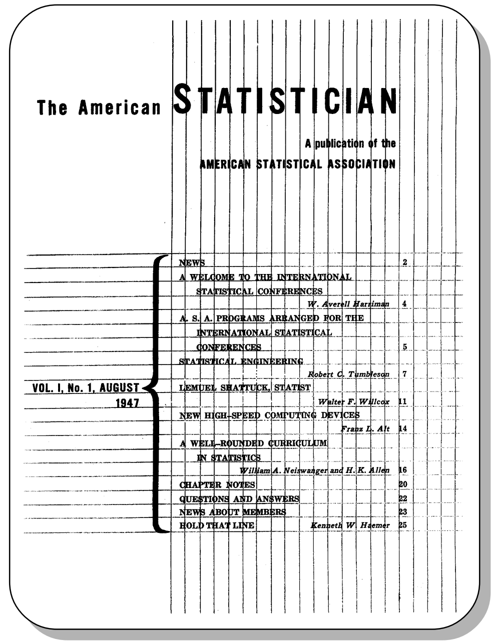
History Corner



The American STATISTICIAN

A publication of the AMERICAN STATISTICAL ASSOCIATION

VOL. I, No. 1, AUGUST 1947

The first issue of TAS *appeared in August 1947.*

t Distribution Centennial

Student's z, t, and s: What if Gosset had R?

James A. HANLEY, Marilyse JULIEN, and Erica E. M. MOODIE

The year 2008 marks the 100th anniversary of the publication of *The Probable Error of a Mean* by William Sealy Gosset, *nom de plume* "Student." Gosset's work and his relationships with the leading statisticians of his day have been considered by several authorities. Despite the extensive documentation, and the seminal nature of the work, modern-day statistics textbooks give him, and this 1908 article, short shrift. Thus, few of today's students—or their teachers—are aware of the "z" statistic whose sampling distribution he actually derived, the mathematical derivation, his simulations to check his work, the material used in the simulations, the table he produced, the "one-line" missing proof supplied by the 22-year-old Fisher (still a student himself) or the subsequent switch, in collaboration with Fisher, from the z to the t statistic. We remind readers of these aspects, and rework his calculations using 21st century computing power. We hope that the next generation of statisticians come to know more about the man and his work than simply that "he worked for the Guinness brewery," and appreciate that not all statistical distributions are derived in a single pass. Research students would do well to use his 1908 article as a model when writing their first statistical article.

KEY WORDS: Biography; Distribution; Extensions; History; Simulation.

James A. Hanley is Professor of Biostatistics in the Department of Epidemiology, Biostatistics, and Occupational Health at McGill University, Montreal, Quebec, H3A 1A2, Canada (E-mail: *james.hanley@mcgill.ca*). Marilyse Julien is a Graduate Student in the Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada (E-mail: *marilyse.julien@mail.mcgill.ca*). Erica E. M. Moodie is an Assistant Professor of Biostatistics in the Department of Epidemiology, Biostatistics, and Occupational Health, Montreal, Quebec, H3A 1A2, Canada (E-mail: *erica.moodie@mcgill.ca*). James Hanley and Erica Moodie acknowledge the support of the National Sciences and Engineering Research Council of Canada (NSERC).

1. INTRODUCTION

The work *The Probable Error of a Mean* (Student 1908) that led to today's t distribution was the lead article in the March 1908 issue of *Biometrika*. Its author was William Sealy Gosset [1876–1937], who—for proprietary reasons—wrote under the pen-name "Student." His Guinness colleague (McMullen 1939) told us a lot about "Student as a man" while E.S. Pearson [ESP] (Pearson 1939) described his groundbreaking statistical work and his links with Karl Pearson and Ronald Fisher. ESP also described how Gosset influenced his own work with Neyman (Pearson 1939, p. 242). R. A. Fisher (1939) wrote

 DOI: 10.1198/000313008X269602

> W. S. Gosset (was. . .) one of the most original minds in contemporary science. Without being a professional mathematician, he first published, in 1908, a fundamentally new approach to the classical problem of the theory of errors, the consequences of which are only still gradually coming to be appreciated in the many fields of work to which it is applicable. The story of this advance is as instructive as it is interesting.

Before turning to this 1908 paper—"one of the seminal contributions to 20th century statistics" (Lehmann 1992)—and our particular interest in it 100 years on, we paraphrase ESP's account of what led up to it. Gosset, who was born in England in 1876, studied mathematics and chemistry at Oxford, and was hired as a staff scientist by Messrs Arthur Guinness Son and Co., Ltd. in Dublin in 1899. His 1904 internal report on "The Application of the 'Law of Error' to the work of the Brewery" (see Pearson (1939) for more details) set out several statistical principles for the practical use of statistical methods. For his report, he relied on two classic texts, those of Airy (1861) and Merriman (1877), but a number of issues not dealt with in these were of concern to him. In 1905 he met with Karl Pearson about three specific problems, one of which became the subject of his 1908 paper:

> I find out the P.E. (Probable Error) of a certain laboratory analysis from n analyses of the same sample. This gives me a value of the P.E. which itself has a P.E. of P.E./$\sqrt{2n}$. I now have another sample analysed and wish to assign limits within which it is a given probability that the truth must lie. e.g. if n were infinite, I could say "it is 10 : 1 that the truth lies within 2.6 of the result of the analysis," *As however n is finite and in some cases not very large, it is clear that I must enlarge my limits, but I do not know by how much* [italics ours].

He spent the 1906–1907 year at Karl Pearson's Biometric Laboratory in London. There, he wrote his first paper (Student 1907) on the sampling error involved in counting yeast cells. His papers on the sampling error of a mean (the subject of this note), and of the correlation coefficient, were published the following year. Cox (2001) commented that these three statistical papers, and the others Gosset published, "have, as has often been remarked, an astonishing freshness and modernity, stemming perhaps from his conciseness and his ability to obtain statistically meaningful results with simple mathematics."

Gosset worked for Messrs Guinness from 1899 to 1935 in Dublin, and 1935 to 1937 in London. Since his death, at age 61, in 1937, the influence of his work and his relationships with leading statisticians of his day have been considered by several authorities. Despite the extensive documentation, and the seminal nature of the work, modern-day statistics textbooks give him, and his 1908 article, short shrift. In what follows, we try to redress this, by re-examining the article in some detail. We hope to attract the attention of younger students by highlighting Gosset's simulations, and how he might have performed them had he had access to today's statistical software. Moreover, we agree with ESP (1939), that

> It is a paper to which I think all research students in statistics might well be directed, particularly before they attempt to put together their own first paper. The actual derivation of the distributions of s^2 and z, or of $t = z\sqrt{n-1}$ in to-day's terminology, has long since been made simpler and more precise; this analytical treatment need not be examined carefully, but there is something in the arrangement and execution of the paper which will always repay study.

2. THEORETICAL DERIVATIONS

2.1 Title and Introduction

In his 50th anniversary review of the two 1908 papers, Welch (1958) focused on Gosset's "p-values with a Bayesian flavor"—and the looseness that was common in the writings (see quotes below) of Gosset and other statisticians of his time—by beginning with a 1892 textbook definition of the term "probable error" [see Verduin (2007) for some earlier uses]. Even though such wording would be even more "statistically incorrect" today, we repeat it merely to explain that the "probable error" studied by Gosset is the *estimated* median deviation of the sampling distribution of the sample mean $\bar{x}$, when $x \sim N(\mu, \sigma)$.

> The probable error of a final result is frequently written after it with the sign ±. Thus, if the final determination of an angle is given as 36" 42' .3 ± 1' .22, the meaning is that the true value of the angle is exactly as likely to lie between the limits thus assigned (that is, between 36" 41' .08 and 36" 43' .52) as it is to lie outside of these limits.

Gosset (p. 1) began by explaining that the "usual method of determining the probability that the mean of the population [μ] lies within a given distance of the mean of the sample [$\bar{x}$], is to assume a normal distribution about the mean of the sample with a standard deviation equal to $s/\sqrt{n}$, where s is the standard deviation of the sample, and to use the tables of the [Normal] probability integral," that is, to assume $\mu \sim N(\bar{x}, s/\sqrt{n})$. But, with smaller n, the value of s "becomes itself subject to increasing error." Sometimes we can use a more reliable external value of s, but "in some chemical, many biological, and most agricultural and large scale experiments," we are forced to "judge of the uncertainty of the results from a small sample, which itself affords the only indication of the variability." Inferential methods for such small-scale experiments had "hitherto been outside the range of statistical enquiry." Rather than merely complain, Gosset did something about it.

> Although it is well known that the method of using the normal curve is only trustworthy when the sample is "large," no one has yet told us very clearly where the limit between "large" and "small" samples is to be drawn. The aim of the present paper is to determine the point at which we may use the tables of the (Normal) probability integral in judging of the significance of the mean of a series of experiments, and to furnish alternative tables for use when the number of experiments is too few.

When we first read his article, we were struck by his crisp preview of each of the ten sections of the paper. We found that E.S. Pearson (1939, p. 221–222) had also been impressed. He suggested that "beginners in the art of composition" follow Gosset's pattern: "first say what you are going to say, then say it and finally end by saying that you have said it."

2.2 The Sampling Distributions of $\{\bar{x}, s\}$ and $z = (\bar{x} - \mu)/s$

Gosset defined s^2 as the sum of squared deviations divided by n, rather than the $n - 1$ (suggested in Airy's textbook) that yields an unbiased estimator of σ^2—a decision influenced by his professor, Karl Pearson. Gosset would have preferred to use $n - 1$: he wrote to a Dublin colleague in May 1907, "when you only have quite small numbers, I think the formula with the divisor of $n - 1$ we *used to use* is better" [italics ours]. Even in 1912, Karl Pearson—still a large-sample person—remarked to him that it made little difference whether the sum of squares was divided by n or $n - 1$, "because only naughty brewers take n so small that the difference is not of the order of the probable error!" (Pearson 1939).

Gosset derived and tabulated the distribution of $z = (\bar{x} - \mu)/s$, rather than $(\bar{x} - \mu)/(s/\sqrt{n})$. He used the distribution of z to make probability statements concerning μ, and thus it may have seemed more natural to him to express the distance between μ and a value of interest, for example, 0, as a multiple of s, in the same way we express an effect size today.

He arrived at the distribution of z in three steps. In section I he derived the first four moments of s^2. He found that they matched those from a curve of Pearson's type III (Wikipedia 2007), and concluded that "it is probable that that curve found represents the theoretical distribution of s^2." Thus, "although we have no actual proof, we shall assume it to do so in what follows." From this, he found the pdf of s by the usual change of variable method; but instead of using technical terms, he put it more intuitively: "since the frequency of s is equal to that of s^2, all that we must do is *compress the base line (axis) suitably*" [italics ours]. Hanley and Teltsch (2006) recently advocated that textbooks emphasize the change of *scale* over the more common change of *variable* terminology.

He promised in his Introduction that in section II "there is shown to be *no kind of* correlation" [italics ours] between $\bar{x}$ and s; presumably he meant that the two are *independent*. His proof is incomplete: he argued that since positive and negative values of $\bar{x} - \mu$ are equally likely, there cannot be correlation between the absolute value of $\bar{x} - \mu$ and s. The result is true in the Normal case, but his argument holds only with "absolute" omitted. As an example of a symmetric distribution where this general reasoning does not hold true, consider $n = 2$ iid Bernoulli(0.5) random variables. However, as noted by Fisher (1925, p. 92), he does show, using the higher moments of the Normal distribution, that the covariance between $(\bar{x} - \mu)^2$ and s^2 is zero.

In section III he derives the pdf of z: he starts with the joint distribution of $\{\bar{x}, s\}$, transforms to that of $\{z, s\}$, and finally integrates over s to obtain $\text{pdf}(z) \propto (1 + z^2)^{-n/2}$. Sections IV and V are devoted to the moments, shapes, and tail areas of the s and z distributions, and to why he only tabled his curve for values of $n \leq 10$ (see Section 4.1).

3. SIMULATIONS AS A "PRACTICAL TEST"

3.1 Gosset's Simulations, Carried out in 1907

An even less well known, but fascinating feature of Gosset's paper is section VI, where the pdf's of s and z "are compared with some actual distributions" in a "practical test of the foregoing equations." Even though he could have used one of the stochastic simulation methods known at that time—for example, dice, spinners, or cards (Stigler 1991)—Gosset, a scientist who worked only on real data, chose instead to use an actual population. He tells us . . .

> Before I had succeeded in solving my problem analytically, I had endeavoured to do so empirically. The material used was a correlation table containing the height and left middle finger measurements of 3,000 criminals, from a paper by W. R. Macdonell (*Biometrika*, Vol. I, p. 219). The measurements were written out on 3,000 pieces of cardboard, which were then very thoroughly shuffled and drawn at random. As each card was drawn its numbers were written down in a book, which thus contains the measurements of 3,000 criminals in a random order. Finally, each consecutive set of 4 was taken as a sample—750 in all—and the mean, standard deviation, and correlation of each sample determined. The difference between the mean of each sample and the mean of the population was then divided by the standard deviation of the sample, giving us the z of Section III. This provides us with two sets of 750 standard deviations and two sets of 750 z's on which to test the theoretical results arrived at.

Macdonell's data, obtained from the Central Metric Office, New Scotland Yard, were reported as a 42×22 frequency table (electronic version at *www.epi.mcgill.ca/hanley/Student*). The 42 rows for the finger lengths correspond directly to the actual measurements, taken to the nearest millimeter, so that the bins are 0.1 cm wide. The height measurements were taken to the nearest 1/8th of an inch by Scotland Yard staff, but grouped by Macdonell into 22 bins, each 1-inch wide and containing those recorded as x and 5/8th inches to $(x + 1)$ and 1/2 inches. The mean (inches) was reported as 65.5355 ± 0.0313, the latter being the probable error, that is, $0.6745 \times sd/\sqrt{3000}$. The standard deviation sd, calculated using $n = 3000$ as divisor and Sheppard's (1898) correction for rounding/grouping was reported as 2.5410 ± 0.0221, the latter being $= 0.6745 \times sd/\sqrt{2n}$.

3.2 Our Simulations, Carried out in 2007

Methods: We first reproduced the means and standard deviations reported by Macdonell. We then repeated Gosset's procedure to create 750 samples of size 4. We occasionally encountered, as did Gosset, a sample where all 4 persons were from the same bin in the original table, so that the calculated s was zero. As Gosset did, we replaced the resulting infinite z value by $\pm$ the largest absolute observed value, depending on the sign of the numerator, and calculated the standard deviation s using

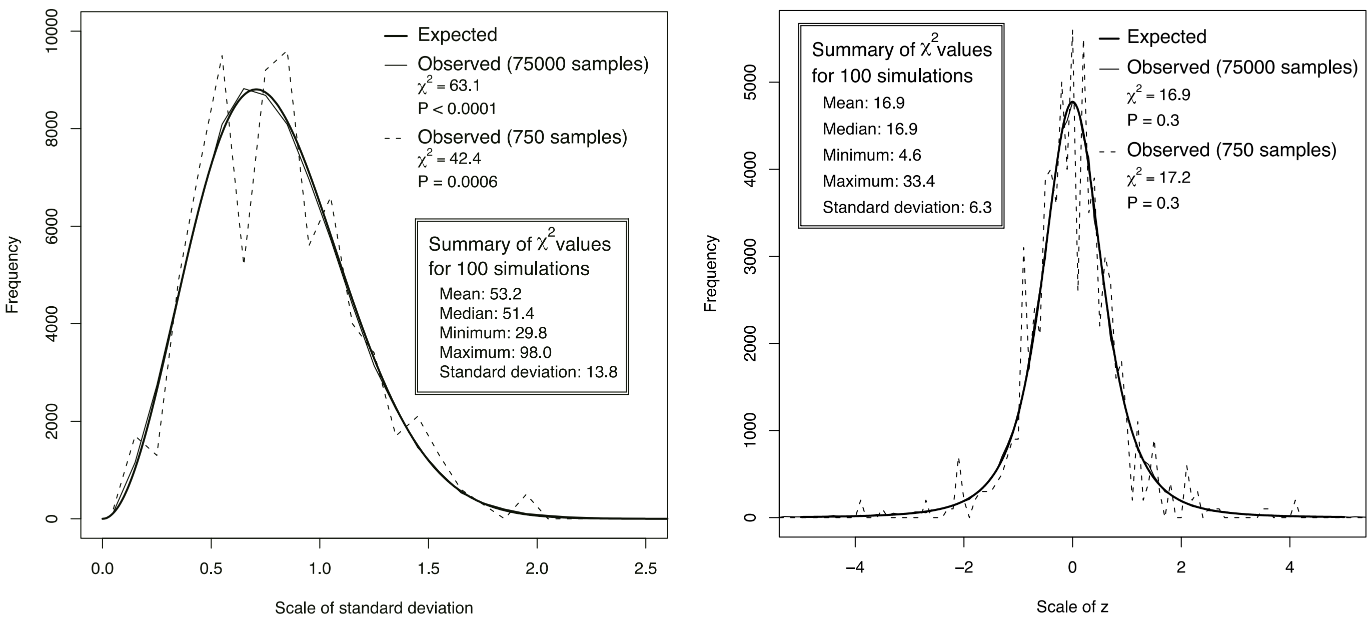


Figure 1. Distributions of s/σ (left) and z (right) in samples of size $n = 4$ from Macdonell's data on heights of 3,000 criminals. Dotted line: (rescaled) distribution of sample statistics obtained from one set of 750 random samples generated by Gosset's procedure. Inset: distribution of 100 chi-square statistics (18 s/σ, 15 z intervals). Thin solid line: distribution of statistics obtained from 75,000 samples of size 4 sampled with replacement from 3,000 heights recorded to the nearest 1/8".

$n = 4$ as the divisor. We calculated a chi-square statistic to measure the discrepancy between the observed and theoretical bin frequencies for the 750 s/σ values, and another for the 750 z values.

We repeated the above procedure 100 times thereby generating 100 different values of the chi-square statistic for the fit of the s, and 100 for the fit of the z, curve. The 100 runs also allowed us to check the repeatability of Gosset's chi-square statistics, and whether the fact that the value of z was infinite (s was zero) in some of his samples may indicate that he had not shuffled the cards sufficiently—as occurred in the 1970 draft lottery (Starr 1997).

We also created a single set of 75,000 samples of size 4, by *sampling with replacement*, and—in the case of height—using the actual precision (nearest 1/8 of an inch) with which heights were recorded by Scotland Yard. The use of the finer height measurements—obtained by adding a random $\pm\{1/16'', 3/16'', 5/16'', 7/16''\}$ to the midpoint of each person's height class—allowed us to judge how much more smooth/accurate Gosset's empirical frequency distribution of s might have been.

Results: For brevity, we present only the results for height, the more coarsely grouped of the measures in Macdonell's table. Complete results, including those for finger length, are available at *www.epi.mcgill.ca/hanley/Student*. In 3 of Gosset's 750 samples, all 4 persons were from the same 1" height class, that is, s was 0. In our 100 repetitions, the numbers of instances in which there were 0, 1, 2, 3, 4, and 5 problematic samples were 21, 41, 17, 16, 4, and 1, respectively, that is, 21 of our runs had situations as or more extreme than his. Thus, his double precautions—very thorough shuffling *and* drawing cards at random—appear to have worked.

He found that the agreement between the observed and expected frequencies of the 750 s/σ's was "not good." He attributed this to the coarse scale of s. His chi-square statistic, summed across 18 bins, was 48.1—just below the median (51) in our series (Figure 1, left), in which values ranged from 30 to 98. The distribution of our 75,000 s/σ values (see Web site) also shows a pattern of large deviations that is similar to those in the table on page 15 of his paper. The frequency distribution of our 75,000 s/σ's obtained by sampling with replacement from heights *recorded to the nearest 1/8"* yielded a chi-square statistic of 63, but was virtually indistinguishable from the theoretical. Thus, Scotland Yard precision and today's computing power would have left Gosset in no doubt that the distribution of s which he "assumed" was correct was in fact correct.

However, for the z's, where the grouping had not had so much effect, he found a "close correspondence between the theory and the actual result:" chi-square statistic, across 15 bins, 12.4. Ours varied from 5 to 33 (median 17); see Figure 1 (right).

4. AFTERMATH

4.1 Remaining Sections of 1908 Article

Gosset was sufficiently convinced by his simulations that in section VII he tabulated the z distribution for $n \leq 10$, and explained its use in section VIII. In section IX, he illustrates the method, using three fully worked examples, all of the "paired-t" type, with n's of 10, 6, and 2. His discussion of the one with $n = 2$ has a strong Bayesian undertone (not to be confused with the loose Bayesian-sounding statements common in his day). He concludes with "an example that comes beyond the range of the tables, there being $n = 11$ experiments," with $\bar{d} = 33.7$ and $s = 63.1$. For this, he uses the approximation

$\Delta \sim N(\bar{d}, s/\sqrt{n-3})$ to arrive at the statement that there is a 0.934 probability "that kiln-dried barley seed gives a higher barley yield than non-kiln-dried seed." His approximation was remarkably accurate: the extended z table he published in 1917, and the `pt` function in R, both yield an exact probability of 0.939.

ESP tells us that "the z-test was used in the brewery at once, but I think very little elsewhere for probably a dozen years." Indeed, in 1922, when Gosset sent Fisher a copy of his new tables, he quipped that "you are the only man that's ever likely to use them!" [cited by Box (1981)]. However, Winkelstein (2004) recently uncovered an "extra-mural" use in a 1912 report.

4.2 Theoretical Confirmation that Distribution of z was Correct

Later in 1912, Ronald Fisher, just graduating from Cambridge, sent Gosset a rigorous and elegant derivation of the z distribution, one that ultimately led Fisher to realize the far wider applicability of variants of Student's z. Gosset asked Karl Pearson, "Would you mind looking at it for me; I don't feel at home in more than three dimensions even if I could understand it otherwise (...) It seemed to me that if it's all right perhaps you might like to put the proof in a note. It's so nice and mathematical that it might appeal to some people." Pearson replied "I do not follow Mr. Fisher's proof and it is not the kind of proof which appeals to me." (Pearson et al. 1990, p. 47). As a result, the proof was only published in Fisher's 1915 paper. In it, Fisher pointed out "that the form establishes itself instantly, when the distribution of the sample is viewed geometrically." (Gosset often teased Fisher about his use of words such as "evidently" and "instantly.") We too did not follow Fisher's cryptic geometric proof until we had read it several times. We have now come to understand and admire "the exceedingly beautiful interpretation in generalised space" which he used to derive the distribution of s (we have placed a "less instant" proof on our Web site). The Editorial in *Biometrika* remarked in passing that Fisher's derivation "shows that for normal distributions there is no correlation between deviations in the mean and in the standard deviation of samples, a familiar fact." It concluded by noting that "the paper by Mr Fisher and the accompanying table (of the distribution of s) more or less complete the work on the distribution of standard-deviations outlined by 'Student' in 1908." The publication of Gosset's extended tables (Student 1917)—from $n = 2$ to $n = 30$—of his z distribution did indeed end the chapter on z. But a new, more extensive and much more important one on t—that took until 1925 to reach publication—was being opened by Gosset and Fisher.

4.3 From z to t

Several authors have described the collaboration between Fisher and Gosset that led to the switch from z to $t = z\sqrt{n-1}$. In a breakthrough paper, Fisher (1925) described the wider uses of t, for example, for two-sample problems and regression coefficients. Student (1925) published his t tables in the journal *Metron*, and Fisher included his own version of the table in his book *Statistical Methods for Research Workers*. Joan Fisher Box (1981) provides the most personal account of this collaboration, including their dealings with Pearson, and the arduous tabulation tasks. Eisenhart (1979) has examined their correspondence from a more technical standpoint; he concluded that the evidence "seems to indicate that the decision to shift from the z to the t form originated with Fisher, but the choice of the letter "t" to denote the new form was due to "Student."" Others who have reviewed Gosset's life and work include Boland (1984) and Lehmann (1999).

5. CONCLUSIONS

Student's 1908 paper has several lessons for those of us who continue to be students of statistics in 2008.

I. To Fisher (1939), "of [Gosset's] personal characteristics, the most obvious are a clear head, and a practice of forming independent judgements." The other was the importance of his work environment: "one immense advantage that Gosset possessed was the concern with, and responsibility for, the practical interpretation of experimental data." And—clearly—Gosset stayed very close to these data.

II. Compared with what Gosset could do, today we can run much more extensive simulations to test our new methods. However, we should ask ourselves which pseudo-random observations are more appropriate: those from perfectly behaved theoretical populations, or those from real datasets, such as Macdonell's. In addition, in light of how he included the three infinite z-ratios, we might re-examine how we deal with problematic results in our runs.

III. Today's students—and their teachers—would do well to heed Pearson's advice regarding writing and communication. Given the decline in the quality of statistical writing, his 1939 message is even more relevant in 2008. We encourage today's students—and their teachers—to read the primary work and other writings of authors such as Galton, Karl Pearson, Gosset, Fisher, E.S. Pearson, Cochran, Mosteller, David Cox, Stigler, and others, not only for interesting statistical content, but also for style.

IV. When the first author of this article was a student, very little of the historical material we have reviewed here was readily available. Today, we are able to obtain it, review it, and follow up leads—all from our desktops—via Google, and using JSTOR and other online collections. Statistical history need no longer be just for those who grew up *B*efore *C*omputers.

[Received July 2007. Revised October 2007.]

REFERENCES

Airy, G. (1861), *On the Algebraic and Numerical Theory of Errors of Observations and the Combination of Observations*, New York: Macmillan.

Boland, P. J. (1984), "A Biographical Glimpse of William Sealy Gosset," *The American Statistician*, 38, 179–183.

Box, J. F. (1981), "Gosset, Fisher, and the t Distribution," *The American Statistician*, 35, 61–66.

Cox, D. R. (2001), "Biometrika: The First 100 Years. *Biometrika*, 88, 3–11.

Eisenhart, C. (1979), "On the Transition from Student's z to Student's t," *The American Statistician*, 33, 6–10.

Fisher, R. A. (1925), "Applications of 'Student's' Distribution," *Metron*, 5, 90–104.

——— (1939), "Student," *Annals of Eugenics*, 9, 1–9.

Hanley, J. A., and Teltsch, D. (2006), "The PDF of a Function of a Random Variable: Teaching its Structure by Transforming Formalism into Intuition," *The American Statistician*, 60, 61–67.

Lehmann, E. L. (1992), *Breakthroughs in Statistics, Vol.II Methodology and Distribution*, New York: Springer-Verlag, pp. 29–32. Johnson, N. L., and Kotz, S., (eds.), Introduction to "Student (1908) The Probable Error of a Mean".

——— (1999), "Student' and Small-Sample Theory," *Statistical Science*, 14, 418–426.

McMullen, L. (1939), "Student as a Man," *Biometrika*, 30, 205–210.

Merriman, M. (1877), *Elements of Least Squares*, New York: Macmillan.

Pearson, E. S. (1939), "Student' as a Statistician," *Biometrika*, 30, 210–250.

Pearson, E. S., R. L. Plackett [ed], and G. A. Barnard [ed] (1990), *Student: A Statistical Biography of William Sealy Gosset*. Oxford, UK: Clarendon Press.

Sheppard, W. F. (1898), "On the Calculation of the Most Probable Values of Frequency Constants for Data Arranged According to Equidistant Divisions of a Scale," *Proceedings of the London Mathematical Society*, 29, 353–380.

Starr, N. (1997), "Nonrandom Risk: The 1970 Draft Lottery," *Journal of Statistics Education 5*. *http://www.amstat.org/publications/jse/* (accessed June 20, 2007).

Stigler, S. (1991), "Stochastic Simulation in the Nineteenth Century," *Statistical Science*, 6, 89–97.

Student (1907), "On the Error of Counting with a Haemacytometer," *Biometrika*, 5, 351–360.

——— (1908), "The Probable Error of a Mean," *Biometrika*, 6, 1–25.

——— (1917), "Tables for Estimating the Probability that the Mean of a Unique Sample of Observations Lies between $-\infty$ and any Given Distance of the Mean of the Population from Which the Sample is Drawn," *Biometrika*, 11, 414–417.

——— (1925), "New Tables for Testing the Significance of Observations," *Metron*, 5, 105–108.

Verduin, K. (2007), "Earliest Known Uses of Some of the Words of Probability & Statistics," *http://www.leidenuniv.nl/fsw/verduin/stathist/1stword.htm*, (accessed June 20, 2007).

Welch, B. (1958), "Student' and Small Sample Theory," *Journal of the American Statistical Association*, 53, 777–788.

Wikipedia (2007), "Pearson Distribution," available online at *http://en.wikipedia.org/wiki/* (accessed July 1, 2007).

Winkelstein, W. (2004), "Vignettes of the History of Epidemiology: Three Firsts by Janet Lane-Claypon," *American Journal of Epidemiology*, 160, 97–101.