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# Cultural imagery and statistical models of the force of mortality: Addison, Gompertz and Pearson

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**Summary.** We describe selected artistic and statistical depictions of the force of mortality (hazard or mortality rate), which is a concept that has long preoccupied actuaries, demographers and statisticians. We provide a more graphic form for the force-of-mortality function that makes the relationship between its constituents more explicit. The 'Bridge of human life' in Addison's allegorical essay of 1711 provides a particularly vivid image, with the forces depicted as external. The model that was used by Gompertz in 1825 appears to treat the forces as internal. In his 1897 essay Pearson mathematically modernized 'the medieval conception of the relation between Death and Chance' by decomposing the full mortality curve into five distributions along the age axis, the results of five 'marksmen' aiming at the human mass crossing this bridge. We describe Addison's imagery, comment briefly on Gompertz's law and the origin of the term 'force of mortality', describe the background for Pearson's essay, as well as his imagery and statistical model, and give the bridge of life a modern form, illustrating it via statistical animation.

*Keywords*: Hazard function; History; Life tables; Mixture models; Teaching

## 1. Introduction

How might we visualize and describe the concept of the force of mortality (hazard or mortality rate) and make more explicit the relationships between its constituent components? What illustrations and imagery have been used by early writers? This essay begins with the mid-20th-century statistical concept of a hazard function, which arose in reliability and failure time analysis in industrial settings, and then describes some cultural and statistical imagery that was used for its human mortality counterpart over the three previous centuries. The paper ends by bringing out of the archives a little known late 19th-century work that combines statistical teaching, imagery and non-standard data analysis.

### 1.1. Early uses of terms 'hazard rate' and 'hazard function'

The hazard function is central to modern survival analysis. Whereas the concept of an instantaneous failure rate is much older, the term 'hazard rate' appears to have been introduced relatively recently. The first mention of it that David (1995) had found is in Davis (1952), where he

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> 'introduce[d] a new probability function which has been found useful in interpreting the physical causes of failure in terms of probability distributions. This function is termed the conditional density function of failure probability with time and is defined as the instantaneous probability rate of failure at time $t$ conditional upon non-failure prior to time $t$'

(page 114). He denoted this 'conditional density function' $f(t)/\{1-F(t)\}$ by $Z(t)$, noted that $Z(t)\,\mathrm{d}t = \Pr(t \leqslant T \leqslant t+\mathrm{d}t | T \geqslant t)$ and pointed out that

> 'the actuarial concept of "force of mortality" is precisely this conditional density function if a human being is considered as a system and death is defined as the failure of the system'.

Later when comparing this function in various failure time models for systems composed of equipment and operating personnel with that for 'human mortality', he noted that

> '[the function] for human mortality is similar in general characteristics to that of the normal theory except that in early life human mortality exhibits a non-zero conditional density. This suggests the rationale that in youth, humans are subjected to a small *death-hazard* rate (force of mortality), but as they age they become increasingly weaker and, therefore, subject to an increasing *death-hazard* rate'

(page 117, italics added).

Zelen (1959), who referenced Davis, seems to be the first to have used the letter $h$. In section 6 of his paper, he tells us that all the results that he established for the exponential distribution can be used for (failure time) distributions having the probability density function of the form

$$f(t) = h(t)\ \exp\left\{-\int_0^t h(x)\,\mathrm{d}x\right\},$$

where 'the function $h(t)$ is simply a non-negative function of the time to failure $t$', and '$h(t)$ is sometimes referred to as the "hazard", "instantaneous rate of failure", or "conditional failure rate" '. Incidentally, in view of the direct link between the survival function and the integral of the hazard function (textbooks provide various proofs of this, e.g. Collett (2003), pages 11–12), one might be tempted to take his statement to mean that the results apply to *all* distributions.

Parzen (1962), page 168, is the first textbook that we know of that defines the hazard function. He denoted it by $\mu(x)$ and called it the *intensity function*, or *hazard function* or *conditional rate of failure function*. Gaver (1963) and Barlow *et al.* (1963) appear to be the first to have included the word hazard in the title of a paper. Gaver used $\lambda(t)$ for the hazard function and $H(t)$ for its integral. Barlow *et al.* (1963) denoted the hazard function by $q(x)$ and tell us that it

> 'is known by a variety of names. It is used by actuaries under the name of "force of mortality" to compute mortality tables. In statistics its reciprocal for the normal distribution is known as "Mill's ratio", ... and in extreme value theory is called the intensity function.'

Klein and Moeschberger (2003) tell us that the

> 'hazard function is also known as the conditional failure rate in reliability, the force of mortality in demography, the intensity function in stochastic processes, the age-specific failure rate in epidemiology, the inverse of the Mill's ratio in economics, or simply as the hazard rate'.

Although the actuaries continue with the letter $\mu$, the letters $Z$ and $q$ have since been replaced by the letter $h$ or $\lambda$—and the argument $x$ by $t$—in the statistical literature, and even in popular Web sources. When we accessed it on March 13th, 2009, *Wikipedia* had defined the *instantaneous hazard rate* as the 'limit of the number of events per unit time divided by the number at risk as the time interval decreases',

$$h(t) = \lim_{\Delta t \to 0} \left\{ \frac{\text{observed events}(t)/N(t)}{\Delta t} \right\}.$$

They have since revised their definition. We find it clearer to omit the somewhat contradictory '$(t)$' that follows the word 'events', and to make both the number of events and the number of people at risk more precise. Seen as a parameter, the hazard involves the expected number of events; the empirical version involves the observed number. Either way, the events occur within an *interval* of width $\Delta t$ centred on or to the immediate right or left of (i.e. that begins or ends at) $t$—we use the left, so that the limit is defined at $t=0$. If we replace the somewhat inexact $N(t)$ by $\bar{N}$, the average number of people at risk during (i.e. contributing follow-up time to) the interval, then the quantity

$$\frac{\text{number of events in } (t, t+\Delta t)}{\bar{N}\,\Delta t} = \frac{\text{number of events in } (t, t+\Delta t)}{\text{persontime in } (t, t+\Delta t)}$$

takes the form of an *incidence density*, a term that was introduced to epidemiology by Miettinen (1976):

> 'Incidence density ("force of morbidity" or "force of mortality")—perhaps the most fundamental measure of the occurrence of illness—is the number of new cases divided by the population-time (person-years of observation) in which they occur'.

Whereas epidemiologists have become comfortable with this term, and with incidence density as a function of $t$, statisticians reserve the term density to describe distributions mathematically and are more comfortable with the concept of *intensity functions* that was introduced by Andersen, Keiding and others in the counting process formulation of survival analysis. However, *in the short term*, i.e. especially if we think in terms of expected numbers, and thus theoretical values, we can speak interchangeably of an incidence density function ID$(t)$, an intensity function $I(t)$, a hazard function $h(t)$ and a force-of-mortality function $\mu(t)$.

### *1.2. Predecessors of, and imagery used for, hazard rate and hazard function*

Whereas statisticians have little difficulty with the idea of a mathematical limit, and thus with the value of the function $h$ or ID at an instant $t$ in time, the concept of an instantaneous ID is more difficult for others. Even if one understands a mathematical limit, the concept of an *instantaneous* force of mortality of, for example, '0.00152 deaths per man-year' or '0.00092 per woman-year' for the instant at which one's internal clock momentarily registers 40:00:00:00 years is not easy to communicate. How might we visualize it and describe it to others, even our mathematically trained students? What statistical images of the hazard function, or force of mortality or incidence density have been used by earlier writers?

Our three selected teachers, all British, are Joseph Addison (1672–1719), an eminent early 18th-century essayist, poet and politician, Benjamin Gompertz (1779–1865), a self-educated mathematician best known among demographers and actuaries, and Karl Pearson (1857–1936), an accomplished historian, 'Germanist' and mathematician who founded the world's first university statistics department. Between them, they used a mix of allegories, physical models and mathematics to visualize and represent the force of mortality, its components and the quantities that are derived from it. Our selection of 'consultants' is not representative, but rather a product of chance connections and circumstances that we shall explain briefly. After describing and commenting on their work, we describe two `java` applets that we have constructed. They are based on the historical imagery but give the force of mortality or hazard function a modern form, illustrating it via statistical animation. They also give form to the $f(t)$ and $1 - F(t)$ functions that define the hazard function.

We begin and end with two striking images, the earlier verbal and the latter visual.

## 2. Addison, 1711

### 2.1. The vision of Mirza

'The vision of Mirza' (Addison, 1711) by Joseph Addison (1672–1719) was published on September 1st, 1711, in the magazine *The Spectator*, which he co-founded. On-line versions of the full text of this allegorical essay are easily found by searching for 'Addison Vision of Mirza', as are reproductions of subsequent drawings based on it. Thus, for brevity, only the central section—with our emphasis added—is reproduced here: Mirza's dialogue with 'one in the habit of a shepherd' who appeared to him when he ascended the high hills of Baghdad to meditate and pray.

> 'He then led me to the highest pinnacle of the rock, and placing me on the top of it, "Cast thy eyes eastward," said he "and tell me what thou seest." "I see," said I, "a huge valley and a prodigious tide of water rolling through it." "The valley that thou seest," said he, "is the Vale of Misery, and the tide of water that thou seest is part of the great tide of eternity." "What is the reason," said I, "that the tide I see rises out of a thick mist at one end, and again loses itself in a thick mist at the other?" "What thou seest," said he, "is that portion of eternity which is called time, measured out by the sun, and reaching from the beginning of the world to its consummation. Examine now," said he, "this sea that is thus bounded by darkness at both ends, and tell me what thou discoverest in it." "I see a bridge," said I, "standing in the midst of the tide." "*The bridge thou seest," said he, "is human life*; consider it attentively." Upon a more leisurely survey of it I found that *it consisted of more than threescore and ten entire arches, with several broken arches, which, added to those that were entire, made up the number to about a hundred*. As I was counting the arches, the genius told me that this bridge consisted at first of a thousand arches; but that a great flood swept away the rest, and left the bridge in the ruinous condition I now beheld it. "But tell me further," said he, "what thou discoverest on it." "*I see multitudes of people passing over it," said I, "and a black cloud hanging on each end of it." As I looked more attentively, I saw several of the passengers dropping through the bridge into the great tide that flowed underneath it; and upon further examination, perceived there were innumerable trap-doors that lay concealed in the bridge, which the passengers no sooner trod upon, but they fell through them into the tide and immediately disappeared. These hidden pitfalls were set very thick at the entrance of the bridge, so that throngs of people no sooner broke through the cloud, but many of them fell into them. They grew thinner towards the middle, but multiplied and lay closer together towards the end of the arches that were entire.*'

The eminent 19th-century epidemiologist William Farr, whom many (e.g. Eyler (1979)) credit with the population-based life table that we know today, was also struck by, and quotes from, Addison's imagery. He does so when writing of 'Uncertainty of individual life and constancy of averages' (Farr (1885), page 455). Although he does not cite any evidence, Farr suggested that Addison's essay 'was probably suggested by Halley's [1693] table'. Then, following up on his comparisons of the sicklier Liverpool, with an average (mean) duration of life of 26 years, and the healthier Surrey, where the average was 45, Farr continued

> 'Our table follows "a throng" of 100,000 that "brake through the cloud" into life at the same moment, and counts them as they step on every arch. It shows, therefore, how many fall through the "hidden pitfalls". The danger is exactly measured. The arches over which sickly multitudes pass, are the same in number as those traversed by a healthy people; but the "trap-doors" and "hidden pitfalls" in their way are twice as numerous, though they can only be perceived by careful observation and counting; while a difference of 26 and 45 "arches" would be obvious to the unassisted eye.'

It is interesting that Addison, although not a statistician, wrote of pitfalls that '*multiplied* and lay closer together towards the end of the arches that were entire'. This would seem to be one of the first descriptions of the smooth-in-time *multiplicative hazards model* that Gompertz parameterized, that Edmonds and Farr exploited (see Section 3) and that Cox (1972) preferred not to specify parametrically.

### 2.2. Animated versions of Addison's bridge

As best we can discern, the original Addison essay was not illustrated. We have developed a `java` applet, which is available at `http://www.biostat.mcgill.ca/hanley/BridgeOfLife/`, that animates his imagery. It shows successive groups of 20 people (blue dots) entering the bridge and marching to the right, and ultimately stepping on a trapdoor (red) and disappearing. Those still surviving continue to walk abreast, with white gaps showing their missing peers. The age-specific numbers of survivors and deaths (proportional to the survival function, $S(t)$, and $f(t)$) are shown in blue and red, and their ratio—the age-specific incidence density or hazard—in black. Also shown in grey is the logarithm of this function of age—its approximate (piecewise) linearity is the focus of the 'law' that was discovered by Gompertz (see below).

The Web site has three examples—the earliest based on the actual experience of the 'cohort' of Swedish females born in 1751; the most recent based on applying the mortality rates of Canadian females in the 3 years 2000–2002 to a hypothetical cohort—i.e. by using a 'current' life table. The middle one applies the mortality data of English males in the 10 years 1871–1880 to a hypothetical cohort, as had been done in the current life table from which Pearson extracted the unconditional frequency distribution for his bridge of life (Section 4).

Users can make their own by replacing the conditional transition probabilities (the $q$-values in the source life table) in the hypertext file. As an example, and to emphasize that a life table refers to a transition from any initial state—desirable or otherwise—to another (absorbing) state, we show a bridge, with just a few arches, based on the time that was spent in pursuit of a doctorate by students in our McGill department over the years 1970–2002.

We next pursue the origins of the evocative term 'force of mortality'. We find them among the early 19th-century actuaries, for whom mental images tended to be secondary to mathematical models and calculus.

## 3. Gompertz, 1825

There is an extensive literature on the history of life tables and the many individuals who have contributed to their development and the concepts behind them. Some of these are John Graunt (1620–1674), Christiaan Huygens (1629–1695), Ludwig Huygens (1631–1699), Edmund Halley (1656–1742), Abraham DeMoivre (1667–1754), Nicholas Bernoulli (1687–1759), Daniel Bernoulli (1700–1782), Antoine Deparcieux (1703–1768), Thomas Simpson (1710–1761), Benjamin Gompertz (1779–1865), Thomas Edmonds (1803–1889), William Farr (1807–1883) and William Makeham (1826–1891). Some of the work has been characterized as mere 'shopkeeper's arithmetic', involving repeated divisions, or use of first differences; other work involved a more formal mathematical approach and applied the methods of calculus to mortality rates by treating them as continuous functions. However, we singled out Gompertz, and his 1825 work (Gompertz, 1825) because of the paradigm shift in his work, and the smooth-in-time or age hazards model that was named after him that we now use in survival analysis. As an eminent actuary, Hooker (1965) wrote 100 years after Gompertz's death,

> 'to the actuarial profession, [his] paper of 1825, in which he propounded his well-known "law" of mortality, marked the beginning of a new era, not merely because his formula was, for several reasons, an enormous improvement on others which had been suggested previously but because it opened up a new

approach to the life table. Previously, the table had been regarded as little more than a record of the number of persons surviving to successive integral ages out of a given number alive at an earlier age; Gompertz introduced the idea that $l_x$ [the survival function] was a function connected by a mathematical relationship with a continuously operating force of mortality.'

Although Gompertz used the word '*intensity*' of mortality in his 1825 article, he did not use '*force*' of mortality. Morabia (2005) has suggested that William Farr was the first to use formally the word 'force' of mortality in his 1838 article 'On prognosis' (Farr, 1838; Hill, 2003a, b; Gerstman, 2003). He wondered whether Farr used the word to characterize a mortality rate because 'he must have been familiar with Newton's 1687 definition of the concept of physical force'. However, Farr merely used the term and we suspect that he took it from Edmonds, a political economist and actuary, and neighbour of Farr's, who strongly influenced Farr's work. Edmonds is the first person whom we know of to define the term: he did so by putting it in italics and in quotes in the first paragraph of Edmonds (1832) (see also the link to an on-line digital version `http://books.google.com`). In chapter 2, where he formally introduced the algebraic expression for his law, he also made a statement that later became the subject of an acrimonious primacy debate within the actuarial profession:

'The honour of first discovering that some connexion existed between Tables of Mortality and the algebraic expression ($a^{b^x}$) belongs to Mr. Gompertz: but, to arrive at this single common point, his course of investigation differs so widely from mine, that appearances will be found corresponding to the reality,—that my discovery is independent of the imperfect one of Mr. Gompertz.'

As many have commented, quite apart from his use of the fluxions notation of his hero Newton, Gompertz's writing and mathematics were not easy to follow. Indeed Edmonds's derivations, using modern notation for differentiation and integration, are much clearer. Edmonds does not give a mathematical rationale for his 'algebraic expression' ($a^{b^x}$) other than to say that it is a law that seems to be borne out by facts—others had noticed this empirical pattern earlier (Woods, 2000). Gompertz does, but we could not translate the imagery in Gompertz's words

'If the average exhaustions of a man's power to avoid death were such that at the end of equal infinitely small intervals of time, he lost equal portions of his remaining power to oppose destruction which he had at the commencement of those intervals'

into its mathematical expression

'then at the age $x$ his power to avoid death, or the intensity of his mortality might be denoted by $aq^x$, $a$ and $q$ being constant quantities'.

Moving—by integration—from there to his 'law' concerning '$L_x$, or the number of persons living at the age of $x$', namely $(S_x =)\ L_x = d(g)^{q^x}$, is easier—but not as easy as Edmonds's way. Gompertz's numerical investigations of the fit of his law to the life tables of the time focused on the survival function $L_x$ rather than on the mortality function. But how did he come by the form $\mu(x) = aq^x$ for the mortality function in the first place? Should we interpret his words to mean that there is a constant external force or threat? Is a man born with a certain power to avoid death, which is progressively exhausted with age? Chiang (2005) has provided a way to see how 'a power to resist death that decreases at a rate proportional to the power itself' can indeed lead to the stated form:

'since the force of mortality $\mu(t)$ is a measure of a person's susceptibility to death, Gompertz used the *reciprocal* $1/\mu(t)$ as a measure of a person's *resistance* to (i.e., power to oppose) death'.

Thus, with $k = \log(c)$, where $c$ is a constant, we can write the 'constant loss of power to oppose' as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{\mu(t)}\right\} = -k\frac{1}{\mu(t)}.$$

This also implies that $\mathrm{d}\mu(t)/\mathrm{d}t = k\mu(t)$, and that $\mu(x)$ has the form $\mu(x) = aq^x$.

Gompertz's main justification for considering his law to be 'deserving of attention' was that 'it appears corroborated during a long portion of life by experience' rather than 'in consequence of its hypothetical deduction'. However, he did add that it

> 'in fact is congruous with many natural effects, as for instance, the exhaustions of the receiver of an air pump by strokes repeated at equal intervals of time'.

In the words of one of our medical colleagues,

> 'the idea of internal decay or loss of power is a correlate of the machine model of pathophysiology with breakdowns and scrapyards being the prevailing metaphors'

(A. Fuks, personal communication).

Gompertz showed the good fit of his law to five life tables of his time, by comparing 'observed' and fitted values of what he called $L_x$. Edmonds and Farr both used different (piecewise) versions of Gompertz's parameteric law for three or four different periods of life, thereby greatly simplifying the calculation of the various quantities, such as age-specific expectation of life, derived from life tables.

We now describe the imagery—again involving a bridge—employed by someone who used his considerable literary and mathematical talents in an essay that deserves to be better known. His graphical depiction helped to illustrate mixture models and competing risks.

## 4. Pearson, 1897

### *4.1. Context for Pearson's The Chances of Death*

Our interest in Pearson's essay was aroused by an image that was commissioned by him and used as the frontispiece to his book *The Chances of Death and Other Studies in Evolution* (Pearson (1897); see also the link to an on-line digital version `http://pds.lib.harvard.edu/pds/view/8144959`). Pearson, Cambridge-trained mathematician, 'Germanist', historian, legal trainee and statistician, as well as founder of the first university statistics department (University College London, 1911), was a polymath. Born Carl Pearson, he formally adopted the Germanic spelling Karl in keeping with his love for a country where he spent several years in the late 1870s. His interest in the numerous cultural and artistic representations of mortality may have been kindled in 1875 when, as an 18-year-old, he had seen the many paintings dating from 1626 to 1635 representing a 'Dance of Death' under the roof of the Spreuer bridge in Lucerne (Switzerland, 2009).

In his essay *The Chances of Death*, which was first delivered as a lecture in January 1895, before the Leeds Philosophical and Literary Society, Pearson described—in words and figures—medieval, cultural images of 'Death as the lawless one, the one who strikes at random'. He provided considerable evidence for the non-random nature of chance (and hence Death) and suggested that we should move away from medieval concepts of chance as something unexplainable. The 'blindness to age' in the dances of Death may have been from the less-age-specific mortality patterns that were observed during plagues, and Pearson may have underemphasized the deterministic nature of these dances (Klein (1997), chapter 2). However, his poetic licence makes a more interesting statistical application.

Pearson does not refer to Gompertz's work, the term 'force of mortality' or to the purposes for which, two centuries earlier, Caspar Neumann collected the Breslau information that was subsequently converted into the first data-based life table by Edmond Halley (Ward (1992), page 69). Pearson was, however, aware of, and possibly influenced by, Addison's essay; indeed, he begins his essay with five sentences from Mirza's description of the bridge. He then proceeds to describe in detail, and to illustrate with reproductions of medieval drawings, some common cultural images of the time. He begins

> 'There is an old German proverb: "Death has no calendar", which taken in conjunction with our English, "Death is no respecter of persons," strongly marks the folk-conception of Death as of one who obeys no rule of time, or of place, or of age, or of sex, or of household. This idea of Death as the lawless one, the one who strikes at random, arose early in mediæval tradition and is represented in the well-known Dances of Death, from the primitive block-book to the finished designs of Holbein. Parallel with this notion of the random character of Death's aim, has run the mind of the folk idea of Chance as that which obeys no rule and defies all measure and prediction.'

The convergence of the concepts of chance and Death came through the folk ideas of the gambler casting dice at a point in time to determine whether Death would pay a visit to a given individual.

### *4.2. Pearson's 'modern' notion of chance and frequency distributions*

Pearson argued as incorrect the notion of chance as something 'which obeys no rule and defies all measure and prediction', and thus devoted the early part of his essay to building up evidence for order and rules in chance distributions by showing four empirical frequency distributions that were obtained in large numbers of repetitions of 'experiments' in his first figure. These involved the number of

(a) red counters out of 10 drawn from a bag of 25 each of four different colours;
(b) hearts in 10 cards drawn from a pack of 52 cards;
(c) dice showing a 5 or a 6 in throws of 12 dice and
(d) heads in tosses of 10 coins.

With these, Pearson introduced the notion of the mode or most frequent outcome. He also used the counter and card drawing examples to emphasize the concept of non-independence of the results of the successive draws and—although his data analysis ignored it—of the importance of such a concept for the mortality curves to be presented: 'frequency of death at later ages must depend on the incidence of death at earlier ages'. Likewise, he used the four experiments to reiterate that, although results of a single trial remain difficult to predict, a large number of experiments do indeed obey a law. Such a law can be represented by using a frequency curve, which has properties of centre, which he described as measured by the mode, mean or median, and of spread as measured by the standard deviation, which Pearson described as the measure of 'concentration of frequency around the mean' itself analogous to the notion of swing radius in the mechanics course that he had taught so often as Professor of Applied Mathematics at University College London. Likewise, he emphasized the notion of skewness of distributions providing a measure of the amount of asymmetry of the frequency curve, defined as the ratio of the deviation of the mean from the mode, to the standard deviation.

Using the examples of the specific physical experiments that were described above, Pearson coaxed the reader into extending the ideas of frequency distributions to other data which arise as a 'product of Nature'. From the physical measurements of sizes of crabs and of human skulls to age scale measurements such as the ages of brides at marriage, he provided examples of frequency curves and hence 'law' in the behaviour of, and events that befall, humans. (He did admit

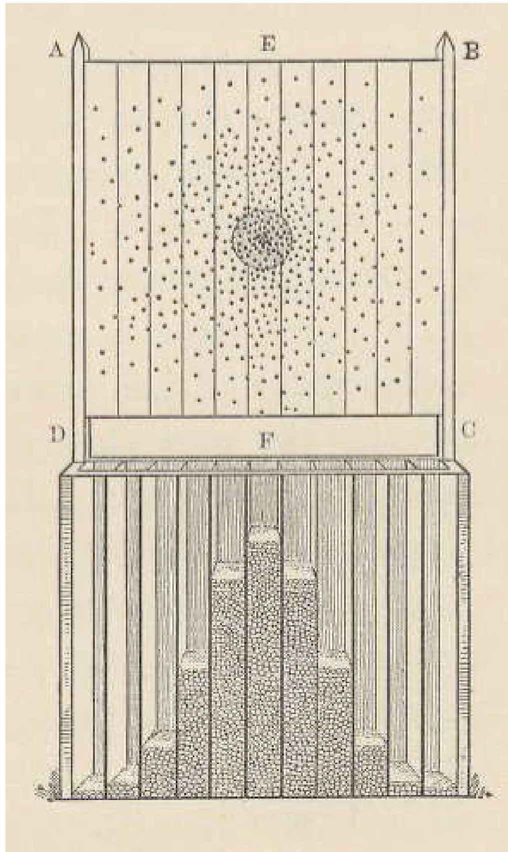


**Fig. 1.** Fig. 5 of Pearson (1897)

that the longer left-hand tail of the distribution of the ages of the brides (of bridegrooms in their 25th year) meant that ‘theory and practice do not, it is true, agree quite as well here as they should do’, but hastened to add ‘and I fear this is due to brides understating their ages, especially in cases where they are older than their bridegrooms’.) From these examples, he argued that ‘... if birth and marriage fall under the general laws of frequency, we may surely expect that death will do so’. Thus we should also be able to represent the force of mortality by using frequency curves and their law.

Before going on to his main task, Pearson gave one last important side illustration, asking us to imagine the distribution of a large number of bullets, each of which, on striking the target, fell ‘without rebound into one of a series of columnar receptacles placed immediately beneath its point of incidence’ (Fig. 1). This distribution might be symmetric, with a standard deviation that reflects ‘the precision peculiar to a marksman or his weapon’. Or it might be that ‘owing to some peculiarity of marksman, weapon or target’ the marksman is ‘more liable to miss badly’ to one side and to generate a skew distribution. Thus we can picture the regularity of the frequency of the ages of death by ‘thinking of Death as a marksman with a certain skewness of aim and a certain precision of weapon’.

Pearson’s figure will remind many of the ‘pins ... disposed in a quincunx fashion’ in the apparatus that was devised a decade earlier by Galton to show his curve of frequency (Galton (1889), page 63). But—possibly because the subject was mortality rather than stature, and because some of the medieval 'Deaths' that were depicted in the dances of Death carried arms—Pearson seems to have been more fascinated with the imagery of marksmen. He was not alone: under the heading 'metaphor and reality of target practice', Klein (1997), who devoted seven pages to Pearson's 1897 essay, tells us that her book 'is about men reasoning on the likes of target practice', that this imagery pervades the thinking and work of natural philosophers and statisticians and that it was not merely a conceptual tool—see Fig. 1.3 on page 11 of Klein (1997). Incidentally, Klein also urged those of us who use the word *stochastic* to appreciate its roots.

### *4.3. Pearson's 'modern' notion of the dance of Death*

In his summary remarks Pearson argued that the 'regularity' that is represented by frequency distributions means that

> 'artistically, we no longer think of Death as striking chaotically; we regard his aim as perfectly regular in the mass, if unpredictable in the individual instance. It is no longer the Dance of Death which pictures for us Death carrying off indiscriminately the old and young, the rich and poor, the toiler and the idler, the babe and its grandsire. We see something quite different, the cohort of a thousand tiny mites starting across the Bridge of Life, and growing in stature as they advance, till at the far end of the bridge we see only the graybeard and the "lean and slippered pantaloon". As they pass along the causeway the throng

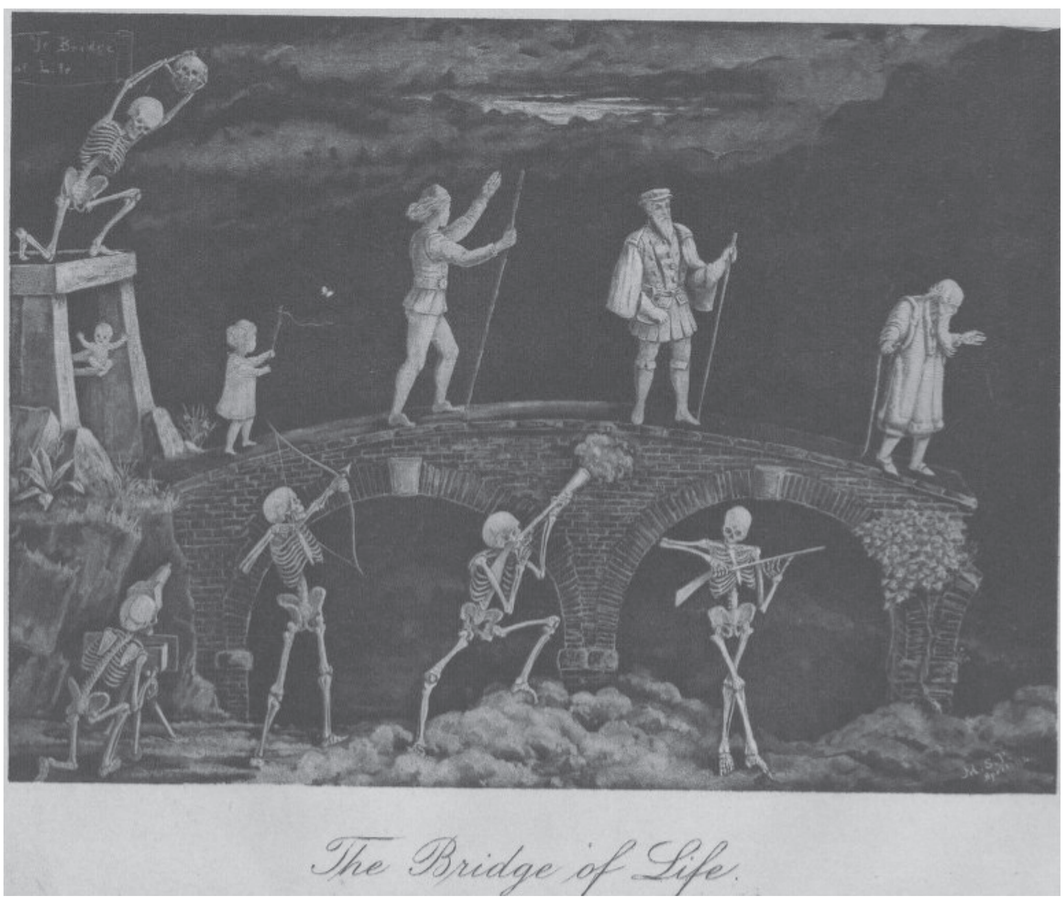


**Fig. 2.** The bridge of life: frontispiece to Pearson (1897)

> is more and more thinned; [a number of] Deaths are posted at different stages of the route longside the bridge, and with different skewness of aim and different weapons of precision they fire at the human target, till none remain to reach the end of causeway—the limit of life.'

As is also illustrated by the many sketches that we found in the Pearson collection in the University College London archives, Pearson went to considerable lengths to convey this imagery to the reader:

> 'It would need a great artist to bring that human procession vividly before the reader. Such alone could not fully realise my dream on the Mühlenbrücke at Luzern of twenty years ago. But I ventured to put the roughest of sketch suggestion before two artists. The one, trained in the modern impressionist school failed, I venture to think, in fully grasping the earnestness of life; the other [his wife Maria Sharpe Pearson], reared among the creations of Holbein, Flaxman, and Blake, shows more nearly the spirit of my dream.'

That imagery which was successfully realized by his wife—the initials 'MSP April '94' are seen at the bottom right, and the book is dedicated to her—is reproduced in Fig. 2. The Pearson collection contains several unsigned sketches of Pearson's bridge of life, one of which links explicitly with the data fitting that is described below, suggesting that some of the imagery was inspired by the data rather than the imagery solely inspiring the model fitting.

Before he described his data and their analysis, Pearson asks us to

> 'imagine a thousand babes to start together along [this] bridge or causeway of life. . . . our cohort shall march slowly across it, completing the journey in something perhaps over the hundred years. . . . At each step Death, the marksman, takes his aim, and one by one individuals fall out of the ranks—terribly many in early infancy, many in childhood, fewer in youth, more again in middle age, but many more still in old age. At every step forward the target alters; those who fall at twenty cannot be aimed at, at sixty, . . . .'

### *4.4. The data, mixture model and fitting*

Pearson analysed data on the age at death of a hypothetical cohort of 1000 English males who were born at the same time based on 1871–1880 death data published in the 1894 edition of Whitaker's almanac (Whitaker, 1894). The raw data, consisting of 100 age-specific bin frequencies, adding to 1000, are shown as a series of connected crosses in Fig. 3 and exhibit a bimodal distribution with many deaths at a young age, which rapidly decreases with a slow increase to older age. Pearson was very much interested in building skew into curves and, where necessary, using mixtures of frequency curves. Although most details of his methods of analysis are omitted from his *Chances of Death* essay, it is evident that Pearson approached the analysis of such data in the manner that he had commonly used at that time, namely by selecting a frequency curve, specifically a mixture of frequency curves, of the type which was most suitable to the form of the shape of the observed data. This is a little different from the approach that is commonly used in statistical inference where a specific form or mixtures of forms (e.g. normal distributions) of frequency distribution is posited and their parameters then estimated. Nonetheless the frequency distribution approach is something that, as noted above, Pearson provided much evidence for in the earlier arguments of his essay.

The specific details of his approach in this instance can be found in an earlier article (Pearson, 1895). Rather than use his commonly employed technique of the methods of moments, a technique that was much influenced by his background in mechanics, he used the method of least squares that had recently been introduced to him by his assistant Udny Yule (Porter (2004), page 241). He proceeded *sequentially* from *right* to *left*—from the marksman aiming at the old to the marksman aiming at infants. For old age mortality, he fitted one of his generalized frequency curves (in this case a reverse gamma distribution—with a longer left-hand tail) to just

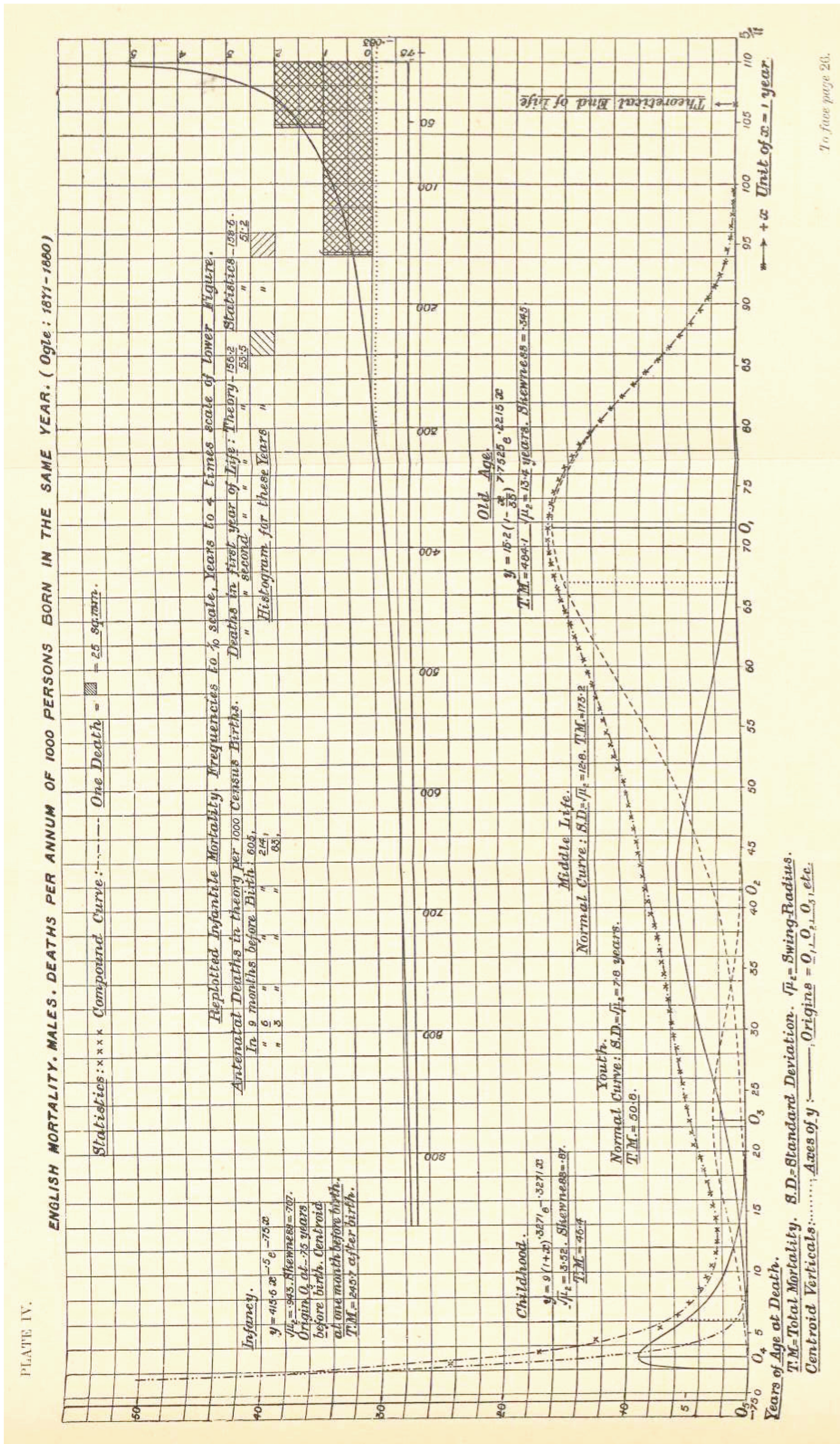


**Fig. 3.** Plate IV in Pearson (1897): frequency distribution of ages at death, along with five components of fitted mixture model (×, frequencies (what Pearson called the 'Statistics', which add to 1000 deaths))

four or five observed frequencies (crosses) at the far right-hand side of the distribution, where the contributions from the competing components would be negligible. He obtained its parameter values by a least squares fit to these four or five numbers. He then subtracted these fitted (and extrapolated to the left) subfrequencies from the still-to-be explained frequencies and proceeded leftwards through the other marksmen, each time using the method of least squares on four or five points at the right-hand edge of the remaining frequencies. To fit the theoretical distribution of ages at death for the leftmost curve, he extended the timescale backwards to begin at conception and to include *antenatal* deaths—an approach that he had adopted for an earlier analysis of deaths from enteric fever.

His unorthodox approach of sequential, mixture curve fitting was pragmatic in nature. Pearson (1895), page 406, told us that

> 'the theoretical resolution of heterogeneous material into *two* components, each having skew variation, is not so hard a problem as might at first appear . . . . If there be more than two components, the equations become unmanageable.'

He noted, however, that if 'the components have rather divergent means, a tentative process will often lead to practically useful results'. The last (15th!) example in Pearson (1895) illustrates this with an example of 'a mortality curve resolved into its chief components', aided perhaps by the new technology of the era—the Brunsviga calculator. Pearson was a proponent of the method of moments in the late 1890s (Porter, 2004) and generally not in favour of the method of least squares.

Pearson did not explain why he chose *five* components, or whether Shakespeare's model of seven ages (to which he referred) was a better fit. Table 1, modelled after Table 2.7 of Klein (1997), describes the five components and the 'marksman' assigned to each: the greater precision of a marksman corresponds to a smaller standard deviation of the corresponding distribution as evidenced by the two leftmost distributions, namely of infancy and childhood. Both of these distributions combine skewness of aim of the marksman.

Pearson's treatment of deaths in infancy is of statistical, political and ideological interest for several reasons. First, the distribution of infant deaths posed an additional challenge in statistical modelling since 'in order to get any fit at all, [it] had to be started very approximately nine months before birth'. Second, he noted that

> 'in England, about a quarter [246/1000 in his fitted model], in France nearer a third than one-quarter of all persons born die as infants'

and suggested that

**Table 1.** Description of Pearson's five-component mixture for frequency of ages at death

| *Pearson's description* | *Descriptors (years)* | | | *Density function* | *Total deaths* | *Marksman's weapon* |
|---|---|---|---|---|---|---|
| | *Mode*† | $\mu$ | $\sigma$ | | | |
| Infancy‡ | $-9/12$ | $-1/12$ | 0.94 | $415.6x^{-0.5}\exp(-0.75x)$ | 246 | Bones of ancestry |
| Childhood | 3 | 6 | 3.52 | $9(1+x)^{0.3271}\exp(-0.3271x)$ | 46 | Maxim gun |
| Youth | 23 | 23 | 7.8 | $N(23, 7.8)$ | 51 | Bow and arrow |
| Middle life | 42 | 42 | 12.8 | $N(42, 12.8)$ | 173 | Blunderbuss gun |
| Old age | 72 | 67 | 13.4 | $15.2(1-x/35)^{7.7525}\exp(0.2215x)$ | 484 | Rifle |

†Pearson imagined the marksman aiming at the mode, which is denoted by $x=0$ in the density function.
‡246 postnatal deaths; 605 antenatal deaths.

> 'a comparatively small reduction in the number of infants who die would be a readier means of checking the decline in the French population than any plan for fostering a higher birth-rate'.

Third,

> 'Our own infantine mortality—amounting to a quarter of all males born—is quite sufficient, however, to occupy our attention, without turning to our neighbours' shortcomings. . . . Thus the sources of this mortality must be sought for in causes common to both periods. These causes must be inevitably associated with parentage. Bad parentage is probably largely the source of this great infantile mortality—bad parentage, showing itself not only physically, but mentally in the want of proper care of the young life, is the one possible cause of death continuous from the antenatal to the postnatal period.'

Thus, whereas he represented the other four marksmen as external forces, 'The marksman Death strikes down the young life with the bones of its ancestry'. Whereas these arguments stem from Pearson's interest in eugenics, Klein (1997) reminded us that decreases in childhood mortality were not achieved by controlling human breeding patterns in subsequent years but by advances in hygiene and nutrition.

The weapons of the marksman of youth and middle life (the bow and arrow and the blunderbuss gun respectively) combined both lower precision (larger standard deviation) of aim, less 'deadliness' (fewer total deaths) and no skewness of aim (both were normal distributions). The rifle of the marksman of old age, aimed at an age of 72 years, combined some skewness to younger ages with the least precision but greatest deadliness (484 of 1000 total deaths). Combined, the mixture represents a series of competing marksmen, namely of competing risks for death, although Pearson's procession from oldest to youngest seems to have the arrows of time statistically reversed.

The reverse gamma distribution that Pearson used for deaths from old age induced an artefactual theoretical upper limit of life of 106.5 years. He did concede that it was a model-based rather than a biologically based limit and in any case

> 'not much stress, however, can be laid on this limit, as an insensible change in the form of the curve sent up, I found, the theoretical end of life some ten years. What is of significance, however, is that a skew curve of this type does give somewhere a theoretical limit to life. The normal chance distribution suggested by Professor Lexis would make the age of Methuselah (969 years) only extremely improbable, not impossible.'

Even with a theoretical limit on the range of the last marksman, it is of course possible that a Methuselah could escape the aim of not just this last one, but also the two marksmen aiming symmetrically but with a theoretically infinite range at youth and middle age.

Overall, the five-part mixture achieves an extremely good fit to the data, no doubt partly because of the smoothness of the overall frequency distribution, which itself was derived from a life table based on 10 years' deaths. The empirical evidence of the frequency-of-death data and such a smooth fit complete Pearson's argument that Death, as governed by chance, is not random, and that there is 'obedience to law' with Death explainable in a population rather than Death acting as the 'lawless one', with no regard for characteristics of the person, in particular for the age of the person. Moreover, through his imagery of the marksman, he suggests that hazards are external to the individual, although age is internal to an individual.

### *4.5. Animated versions of Pearson's bridge of life*

The Web site `http://www.biostat.mcgill.ca/hanley/BridgeOfLife/` contains a `java` applet that we have developed to animate Pearson's imagery. It shows successive people (blue stick figures) entering the bridge and marching to the right, ultimately being struck by one of the five marksmen (to distinguish them better, shots from two of the marksmen are shown in black rather than red) and disappearing, leaving blank spaces in the procession.

The example is based on the mortality data of English males in the 10 years 1871–1880 and is derived from the same current life table that Pearson used for his bridge of life (Section 4.4). Users can supply their own five hazard functions. The programming of the frames, and the decisions on how to have the collisions reproduce the observed data, would be an instructive exercise for statistics students, since it may not be immediately obvious how to go from Pearson's five frequency distributions to the rates and ranges at which the marksmen fire.

## 5. Discussion

Our three teachers have presented a range of allegorical, conceptual, mathematical and statistical models of the force of mortality and quantities derived from it (namely the frequency-of-death distribution and the survival distribution). Table 2 summarizes the contributions of our three teachers in our quest to give a graphic form to the force of mortality and quantities derived from it. Imagery was used by all three: both Addison and Pearson suggested that the force of mortality operated externally to the individual. Gompertz's imagery, invoking the notion of the 'body wearing out', was of forces acting internally.

The bridge in Addison's allegorical vision of Mirza and the imagery in the dances of Death clearly influenced Pearson. He 'modernized' the bridge of life, inspired by the model fitting of the frequency distribution of the ages of death (the numerator of the force of mortality (hazard function)) by using the imagery of marksmen rather than the 'trapdoors' of Addison. In both cases, the axis of the imagery was that of age. Addison's imagery conveys a 'mass' of people moving through life over a river, whereas Pearson's imagery, as realized by his wife, conveys the bridge of life for a single individual passing through the five phases of life (as determined by his mathematical fitting of the mortality data of Fig. 2) aimed at by marksmen standing on solid ground.

Whereas Addison relied on words, Pearson and Gompertz both developed statistical models for the two components whose ratio comprises the force of mortality. Even though the 'data' in the almanack consisted of the two columns 'Of 1,000,000 born, the number *surviving* at the end of each year of life' and 'Mean after-lifetime (expectation of life)', Pearson addressed only the numerator, the frequency distribution of age of death, which he obtained by successive subtractions. Gompertz began with a mathematical form for the force of mortality itself and from it derived the form of the survivor function. Both used grouped data from existing life tables. We have not studied when it was that actuaries made the transition from 'mortality' tables to 'life' or 'vitality' tables, or how often contemporary biostatisticians focus on the hazard function

**Table 2.** Summary of the contributions of our three teachers

| *Teacher* | *Allegory* | *Conceptual model* | *Statistical model* |
|---|---|---|---|
| Addison, 1711 | Yes | External—trapdoors | — |
| Gompertz, 1825 | — | Internal—body wears out | Smooth hazard or survival function |
| | — | External—marksman | Mixture of frequency functions |

rather than the survival function. Readers are referred to the second chapter of Klein (1997) for further discussion of Pearson's treatment of time.

The concept of the force of mortality is the human mortality forerunner of the item failure concept that led in the mid-20th century to the hazard function in the context of industrial life testing. Ironically, the term hazard has now been widely adopted for the study of life events in humans, replacing, in biostatistical texts at least, any use of the term force of mortality, hinted at by Gompertz, and defined by Edmonds, a term which our research suggests is one of the oldest synonyms for the hazard function. Although not historians, we have used the powerful artistic, cultural imagery and statistical depictions in the work of our three selected historical teachers, as well as modern (electronic) computers to trace and animate the concept of the force of mortality and its constituents. In so doing we hope that we have provided a more graphic form for the force of mortality and made more explicit the relationship between its constituent components, to the benefit of students and statistical practitioners alike.

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For getting us started on this work, we thank J. Caro: before we happened on Addison (1711) he related to us how, sometime in the 1990s, Olli Miettinen—unaware of Addison's essay—had illustrated the concept of 'incidence density' to him by using the image of an army marching over a bridge; every so often a spontaneously occurring hole in the bridge swallowed up some of those marching over it. For allowing us to connect Addison's essay—which we had come across in the meantime—with the work of Pearson, we thank J. Olshansky: at the Joint Statistical Meetings in Salt Lake City in 2007, he began his talk 'In search of the Law of Mortality' with a picture of Karl Pearson's graphical depiction of the bridge of life. For their various contributions, we thank A. Burchell, D. Bellhouse, A. Chiolero, H. David, D. Elbourne, A. Fuks, R. Kidman, S. Kovalchik, W. Mietlowski, O. Miettinen, A. Morabia, M. Pagano, E. Parzen, O. Schieir, I. Shrier and M. Zelen. For their encouragement and comments, we thank the many others who helped us along the way.

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