

Cultural imagery and statistical models of the force of mortality

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In order of appearance...

- 1950-1976: hazard function and incidence density
- 1711: allegorical essay, non-mathematical
- 1825 & 1832 : intensity/force of mortality
- 1897: imagery back to 1400's; mixtures of pdf's
- 2009: computer animations

Hazard rate / Hazard function / Force of mortality

- Central to modern survival analysis / epidemiology.
- Other names [Barlow (1963), Klein & Moeschberger (2003)]
 - Demography/Actuarial: “Force of mortality” → life tables
 - Economics: “Mill’s ratio” → reciprocal for normal distr’n
 - Statistics: “Intensity function” → extreme value theory
 - Statistics: “Conditional failure rate” → reliability
 - Epidemiology: “Age-specific failure rate”
- Miettinen (1976): (short-term) “Incidence density”

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- Barlow et al. (1963)

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$$h(t) = \lim_{\Delta t \rightarrow 0} \frac{\text{observed events}(t)/N(t)}{\Delta t}.$$

- Cox, 1972: “age specific failure rate”:

$$\lambda(t) = \lim_{\Delta t \rightarrow 0+} \frac{\text{pr}(t \leq T < t + \Delta t \mid t \leq T)}{\Delta t}.$$

Our edits of Wiki definition of hazard rate

- Omit somewhat contradictory '(t)' that follows word 'events'
- Make '#events' & '#s of persons at risk' more precise.
- Distinguish **parameter** (exp'd #) & **statistic** (obs'd #)
- Events occur within *interval* of width Δt
- $N(t) \rightarrow \bar{N}$, average # persons at risk during interval

$$\frac{\text{no. events in } (t, t + \Delta t)}{\bar{N} \times \Delta t} = \frac{\text{no. events in } (t, t + \Delta t)}{\text{Person-Time in } (t, t + \Delta t)}$$

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Incidence density function $ID(t) \longleftrightarrow h(t)$ hazard function.

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- “This suggests the rationale that in youth, humans are subjected to a small **death-hazard** rate (force of mortality), but as they age they become increasingly weaker and, therefore, subject to an increasing **death-hazard** rate”.

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- Gaver Technometrics 1963 and Barlow Ann. Math. Stat. 1963,
 - First to have used word hazard in title of article ???
 - Gaver uses $\lambda(t)$ for hazard function and $H(t)$ for its integral.
 - Barlow et al. denote hazard function by $q(x)$

Visualizing the force of mortality: describing it to others

- Statisticians comfortable with concept of mathematical limit
- *Instantaneous* force of mortality more difficult for others
 - '0.00152 deaths per man-year at 40th birthday'
 - '0.00092 per woman-year at 40th birthday'
- How can we describe/visualise the force of mortality and quantities derived from it?
- How to give **human form** to components of $h(x) = \frac{f(x)}{1-F(x)}$?
 - 18th and 19th century teachers
 - 21st century computer animation

The force of mortality as seen historically

- Three British teachers
 - Joseph Addison(1672-1719)
 - early-18th century essayist/ poet/ politician
 - Benjamin Gompertz (1779-1865)
 - self-educated mathematician/ demographer / actuary
 - Karl Pearson (1857-1936)
 - historian/ 'Germanist'/ mathematician
 - founder of world's first university statistics department
- Visualize & represent force of mortality / derived quantities
 - allegories
 - physical and conceptual models
 - mathematics & statistical analysis

Force of mortality as seen by contemporary epidemiology teacher

Question: How do you visualize incidence density?



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- Incidence density: number of persons swallowed up divided by, say, the total number of soldier-steps, or soldier-meters or soldier-minutes.
- Incidence density can vary as a function of distance from beginning of the bridge.

Force of mortality in *The Vision of Mirza*

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THE SPECTATOR.

*Interdum speciosa locis, morataque recte
Fabula nullius Veneris, sine pondere & Arte,
Vulgnis oblectat populum, meliusque moratur,
Quam versus inopes rerum, nugæq; canora.* Hor.

The Spectator, June 7. 1711.

IT is the Custom of the *Mohometans*, if they see any printed or written Paper upon the Ground, to take it up and set it aside carefully, as not knowing but it may contain some piece of their *Alcoran*. I will confess there is a kind of the *Mahometans* in me, that I cannot forbear looking into every Printed Paper which comes in my way, unless it should be a plain Almanack. It may appear, for as no *Mohometan*, in the world, I see not a Volume of Things, however it may set his Wits on fire, unless first or last, he scold, a Man may sometimes meet with very occasioned Names in a Piece of Tragedy. I have lighted my Pipe more than once with the Writings of a French, and know a Crowd of men who, for many several Years, but converted the Titles of a Man of Quality into a kind of Prayers for his Castle-Build. I remember, in particular, after having read over a Piece of an Eminent Author on a *Yiddish*, I met with several fragments of it upon the new Repertory, which had been employed in Cards and Croquet, and by the mean obtained to be sold in a little *Leguery*. I once met with a Page of Mr. *Raue* under a *Chevalier* Poem, Whether or not the *Public-Coach* had made use of it through Chance, or Wastery, for the infinite of our *Superstitious Fables*, I know not; but upon one Perusal of it, I considered it good as Men of the Author's Party, that I bought the whole Book. I have often very much profited by such accidental Readings, and have sometimes found very Curious Pieces and other out of Print, or not to be met with in the Shops of our *London Book-Sellers*. For this Reason, when my *Trinitarian* is Surveyor of my Library, my necessary much dependence upon the Shelf of *Feller's*, two long *Band-bears* standing against among my Books, all I do, than do not

they are both of them filled with deep Tradition and divine Literature, I might likewise mention a *Paper-Kiss*, from which I received great Improvement; and a *Five-Cards*, which I would not challenge for all the *Devotions in Great Britain*. The very insipid *Temper*, or other superfluous Humour of paying into all Kinds of *Deities*, with my natural Aversion to *Leguery*, give me a great deal of Employment when I enter my House in the Evening, for I can't, for my *Throat*, have a *Rock* before I have thoroughly studied the *Writings* of it, and executed the several printed Papers which are usually published upon them. The last Piece that I met with, upon this Occasion, gave me a most exquisite Pleasure. My Reader will think I am not honest, when I confess here that the *Piece* I am going to speak of, was the old Ballad of the *Tom-Gardener* in *Scott's* Tale, which is one of the *English Songs* of the Common People, and has been the Delight of most *Englishmen* in some Part of their Age.

This Song is a plain simple Copy of Nature, the Effect of all the *Impartial Observations* that the Tale of it is a very *English* Story, and therefore pleases for another Reason, but because it is a Copy of Nature. There is a certain Simplicity in the Verse, and yet, because the Sentences are natural and unaffected, they are able to move the Heart of the most polite Reader with several Meetings of *Humour* and *Companion*. The Successive growth of the *Story*, and the *Style* itself would have reached upon me, had the *Story* been told by the *Divine Poet*. For which Reason the whole Narrative has descending in it very moving, and interesting the Author of it, (whenever he was delivered it in such an *English* Poem, and proceeded of *Englishmen*, that the *Story* was just of a world.

Force of mortality in *The Vision of Mirza*

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Fabula nullius Veneris, sine pondere & Arte,
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Quam versus inopes rerum, nugæ, canora. Hor.*

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IT is the Gallies of the *Mohometans*, if they be any printed or written Paper upon the Ground, to take it up and set it aside as idle.

It, as not knowing yet it may contain some pieces of their *Alcoran*, I will consult them all, and of the *Mughals* is one, that I cannot forbear looking into every Printed Paper which comes in my way, unless it should be otherwise designed. It may appear; for as no *Mohar* Author, in the ordinary way, writes of these Things, I have to what use his Works may, some time or other, be useful. A Man may sometimes meet with very occasioned News in a Paper of Commerce. I have lighted my Eye more than once with the Writings of a Frenchman, and know a Friend of mine who, for three several Years, has converted the History of a Man of Quality into a kind of *Præcept* for his Castle-Bride. I remember, in particular, after having read over a Piece of an *Emment* Author on a *Yelberty*, I met with several fragments of it upon the same Repetition, which had been employed in *Devils and Cuckers*, and by the mean obtained to be sold in a *Devils* Copy. I came not with a Page of Mr. *Ranger* under a *Civilian* Pen, Whether he set the *Public* Cash had made use of it through *Gloucester*, or *Wentworth*, the Justice of the *Supremacy* *Præcept*, I know not; but upon the Perusal of it, I considered it good as *His* of late Author's Theory, that I bought the whole Book. I have often very much profited by such accidental Readings, and have sometimes found very Curious Pieces that are taken out of Print, or not so far with in the Shops of our *London* Book-Sellers. For the Reason, when the *Printers* take a Survey of any Library, they usually make their Observations upon the Shelf of *Folio's*, two long Boards being standing upright among my Books, all I see, from the out

they are both of them filled with deep Tradition and solid Literature. I might likewise mention a Paper like, from which I have received great Improvement; and a *His* Copy, which I would not challenge for all the *Diversions* of *London*. The very Impulsive Temper, or other supposition, the most of paying into all Kinds of its State, with my natural Aversion to *Liquors*, give me a great deal of Employment when I enter my Head in the Country for I can't, for my Health, have a Room before I have thoroughly studied the State of it, and executed the several printed Papers which are usually published upon them. The last Time that I met with such a *His* Copy, gave me a very singular Pleasure. My Reader will think I am not simple, when I mention here that the *Print* I am going to speak of, was the old Edition of the *Two* *Glosters* in *1707*, which is one of the *Earliest* Songs of the *Common* People, and has been the Delight of such *English* men as have had of both Ages.

This Song is a plain simple Copy of *Nature*, dispassion of all the *Impassioned* *Dispassion'd* *See*, The Tale of it is a pretty *English* Story, and therefore pleases for its own Reason, and because it is a Copy of *Nature*. There is civil and polite, Simplicity in the Verse, and yet, because the Sentiments are natural and unaffected, they are able to move the Heart of the most polite Reader with several Meetings of *Humour* and *Companion*. The Successive good use of the *Simple*, and one *See* in *Every* *Thought* would have made some, but the *See* *See* seems told by the *Devise* *Foot*. For which Reason the whole *Narrative* has something in it very moving, notwithstanding the Author of it, whoever he was, has delivered it in such a simple *Style*, and good use of *English*, that the young set just of a world



Poets' Corner <http://1poet.org>

Joseph Addison

(1672-1719)

Imagery in *The Vision of Mirza*

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BRIDGE OF HUMAN LIFE

Allegorical essay

Sept. 1, 1711

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Multitudes of people passing over it

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- **multiplied and lay closer together** towards the end

Allegorical essay

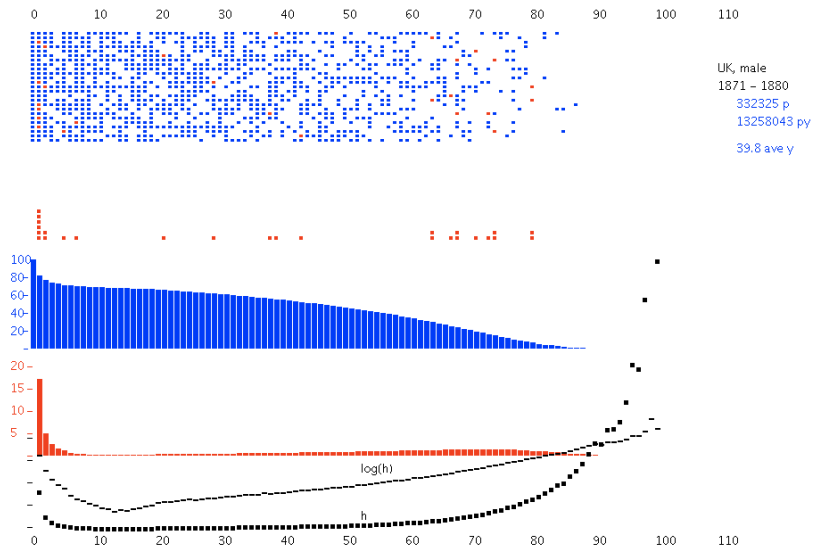
Sept. 1, 1711

Animated Bridge of Human Life

<http://www.biostat.mcgill.ca/hanley/BridgeOfLife/>

WARNING: Moving images shown

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Lifetables; statistical laws and regularity

Some contributors

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'shopkeeper's arithmetic'

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* mortality/survival rates as continuous mathematical functions.

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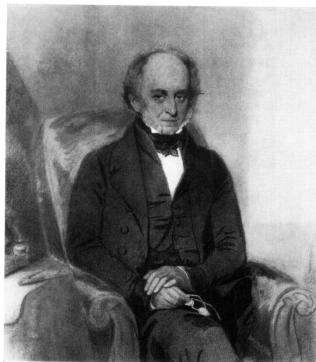
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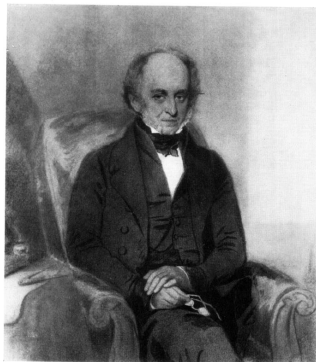


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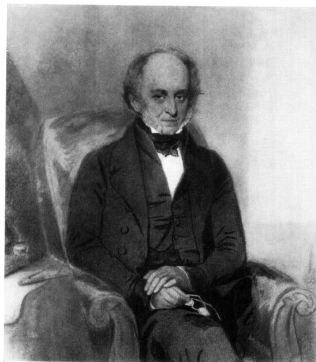
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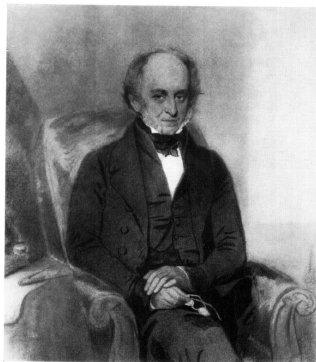


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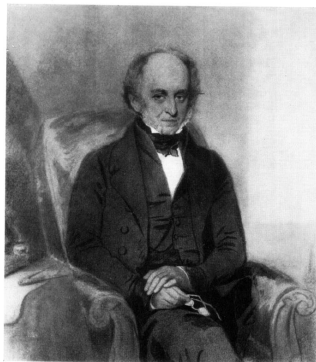
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“ [his] paper [...] opened up a new approach to the life table. Previously, the table had been regarded as little more than a record of the number of persons surviving to successive integral ages out of a given number alive at an earlier age; Gompertz introduced the idea that l_x [the survival function] was a function connected by a mathematical relationship with a continuously operating force of mortality.”

P. F. Hooker. *J. Inst. Actuaries* (1965).

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BENJAMIN GOMPERTZ, 1779–1865

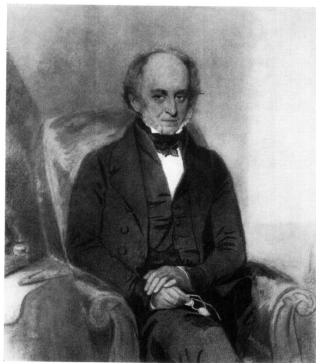
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Imagery: ‘power of man to avoid death’

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 - in ‘*italics and in quotes*’ in first paragraph of his **1832 book**
 - first person we know of to define as *force* of mortality

LIFE TABLES,

FOUNDED UPON

- 33.

THE DISCOVERY

OF

A NUMERICAL LAW

REGULATING THE

EXISTENCE OF EVERY HUMAN BEING:

ILLUSTRATED BY

A NEW THEORY

OF THE

CAUSES PRODUCING HEALTH AND LONGEVITY.

By T. R. EDMONDS, B.A.

LATE OF TRINITY COLLEGE, CAMBRIDGE:

AUTHOR OF

"PRACTICAL MORAL AND POLITICAL ECONOMY."

LONDON:

PRINTED FOR

JAMES DUNCAN 37, PATERNOSTER ROW;

AND MAY BE HAD OF THE AUTHOR,

45, SOUTHAMPTON ROW, RUSSELL SQUARE.

M.DCCC.XXXII.

observed relation of Dying to Living, in given intervals of age. In constructing a Table of Mortality, the ordinary problem for solution is,—given, this relation for large intervals of age; required, to deduce and interpolate the relation of Dying to Living, corresponding to small intervals of age. In all Tables which have hitherto been published, this relation for annual intervals is continually varying. Now it is manifest, that the same principles which have led to the conclusion, that the variation is continued and *annual*, must lead to the conclusion, that the variation is monthly, and also to the conclusion, that the variation is diurnal, and even *momental*. It may be assumed, therefore, that all Tables of Mortality represent the relation of Dying to Living as changing continuously,—that this relation is never the same for any two successive instants of age. I have used the term “*force of mortality*,” to denote this relation at any definite moment of age. It would evidently be improper to use this term to express the relation of Dying to Living in yearly intervals of age; for the force of mortality at the beginning, at the middle, and at the end of any year of age, are all different.

Edmonds' (and our) use of concept of a '*person-year*'

- “The force of mortality at any age is measured by the number of deaths in a given time, **out of a given number constantly living**.

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- Can use Poisson distr'n to derive $S(t) = \exp\{-\int_0^t h(u)du\}$
 - $\mu = \int_0^t h(u)$: *expected* number of events if always 1 (not necessarily *same*) individual at risk for the full $(0, t)$ interval.
 - $h(u)$ will vary over $(0, t)$, but can still view integral as the expected value of an (infinite) sum of Poisson r. v.'s.

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Modern-day notation: **$\mu(x)$** or **$h(\text{age})$** :

$$h(\text{age}) = h_0 e^{\beta \text{ age}} ; \log[h(\text{age})] = \beta_0 + \beta \text{ age}$$

Gompertz translated, with help from Chiang

- Does Gompertz envision constant external force or threat?
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- How can 'a power to resist death that decreases at a rate proportional to the power itself' lead to the stated form?

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'constant loss of power to oppose'

$$\rightarrow \frac{d}{dt} \left\{ \frac{1}{\mu(t)} \right\} = -k \times \left\{ \frac{1}{\mu(t)} \right\}$$

$$\rightarrow \frac{d}{dt} \mu(t) = k \times \mu(t)$$

$$\rightarrow \mu(x) \text{ has form } aq^x. \quad \rightarrow L_x = d \times (g)^{q^x}.$$

Karl Pearson (1857-1936)

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.....|.....|.....|.....|.....|.....|.....|.....|.....+
.1860.....1870.....1880.....1890.....1900.....1910.....1920.....1930.....1940

Mediaeval imagery: the Dance of Death



Under roof of
Spreuer Bridge in
Lucerne, 67
paintings dating
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Mediaeval imagery: the Dance of Death



Death, represented as skeleton or as “Great Reaper” urges everybody to dance with him, i.e. to die. **Death makes no difference between old and young, churchmen and laymen, ...**

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[http://travelguide.all-about-switzerland.info/
lucerne-spreuerbridge-dance-death.html](http://travelguide.all-about-switzerland.info/lucerne-spreuerbridge-dance-death.html)

rich...

beautiful...

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Seen by Pearson in 1875; motivation for his 1897 essay

Pearson's essay *The Chances of Death*, 1897

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THE CHANCES OF DEATH

AND

OTHER STUDIES IN EVOLUTION

BY

KARL PEARSON, M.A., F.R.S.

PROFESSOR OF APPLIED MATHEMATICS IN UNIVERSITY COLLEGE, LONDON,
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WITH ILLUSTRATIONS

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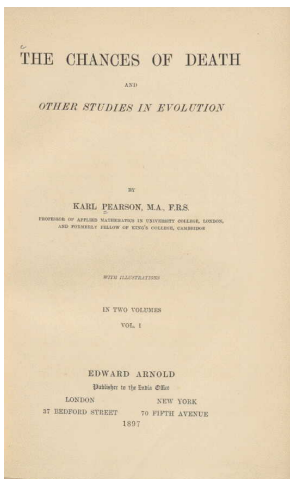
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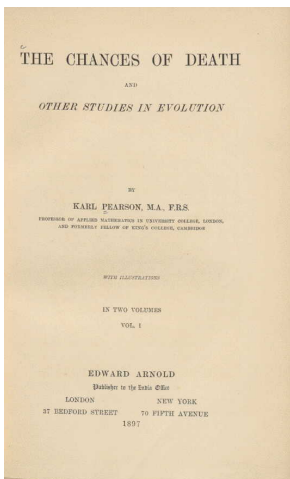
Pearson's essay *The Chances of Death*, 1897

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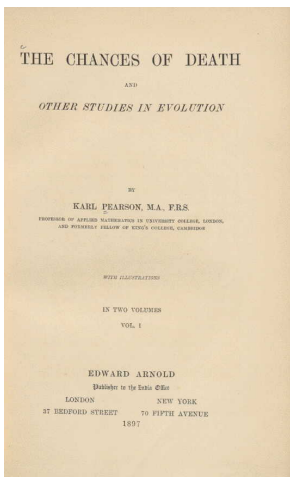
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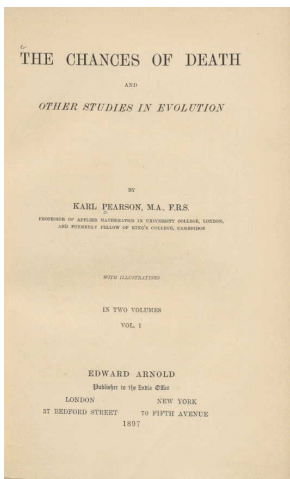


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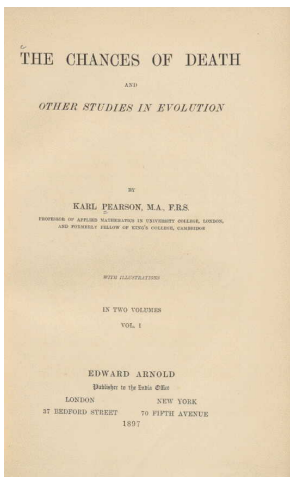


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- "Earliest Dance of Death of which fragments are still preserved to us, appears to be that on the wall of the cloister of a former nunnery at Klingenthal, near Klein-Basel; [...] language and costume of the designs suggest the first quarter of 14th century"

www.dodedans.com/Ebasel-klingental.htm

www.dodedans.com/Eindex.htm

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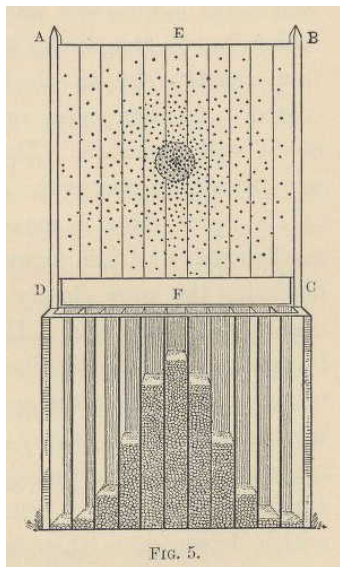
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 - \Rightarrow Death also governed by chance distributions, hence by 'obedience to law'

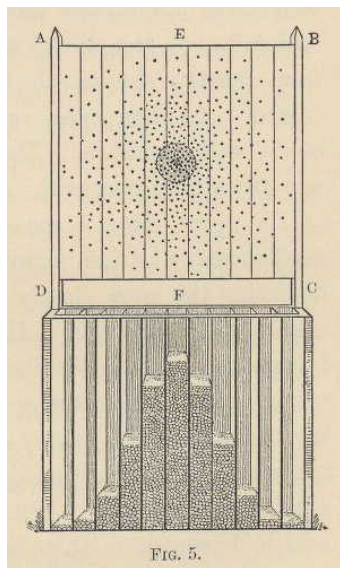
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- Pearson a fan of frequency distributions
 - Data on distributions of deaths

From visual to statistical: Imagery of the marksman

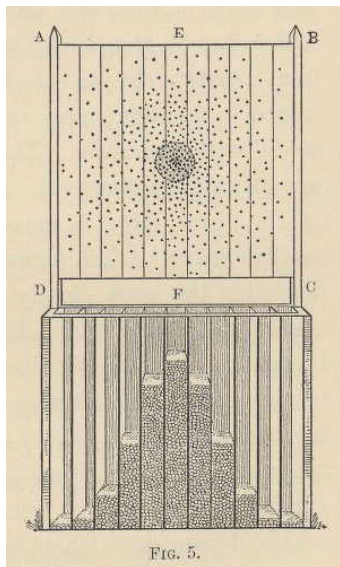


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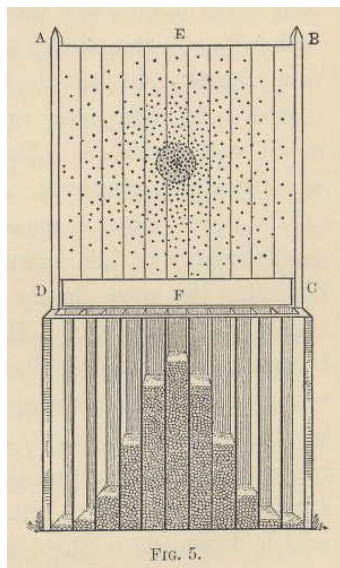
- Marksman fires bullets at target

From visual to statistical: Imagery of the marksman



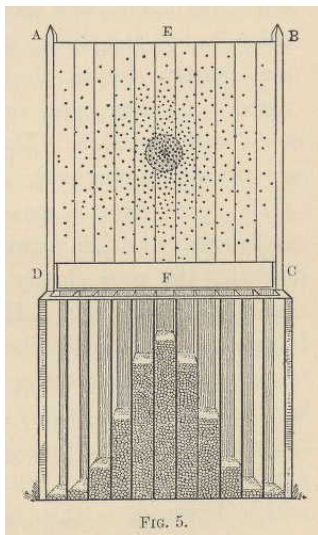
- Marksman fires bullets at target
- Standard deviation: 'precision peculiar to a marksman or his weapon'

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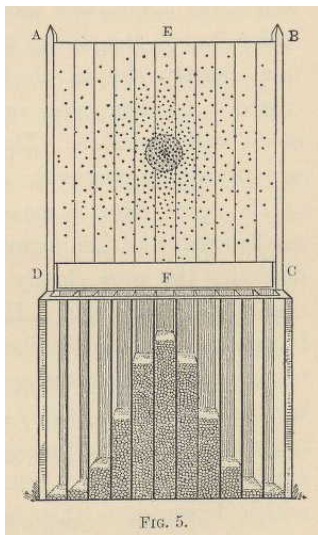
- Marksman fires bullets at target
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- Skew: peculiarity of marksman liable to miss more to left or to right of target

From visual to statistical: Imagery of the marksman



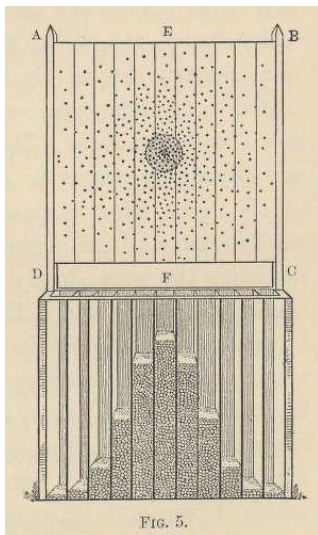
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From visual to statistical: Imagery of the marksman



‘Our ancestors were correct in supposing the frequency of death to be a chance distribution, but we now know that such a distribution follows regular laws, and this regularity we are able to picture to ourselves by thinking of Death as marksman with a certain skewness of aim and a certain precision of weapon’

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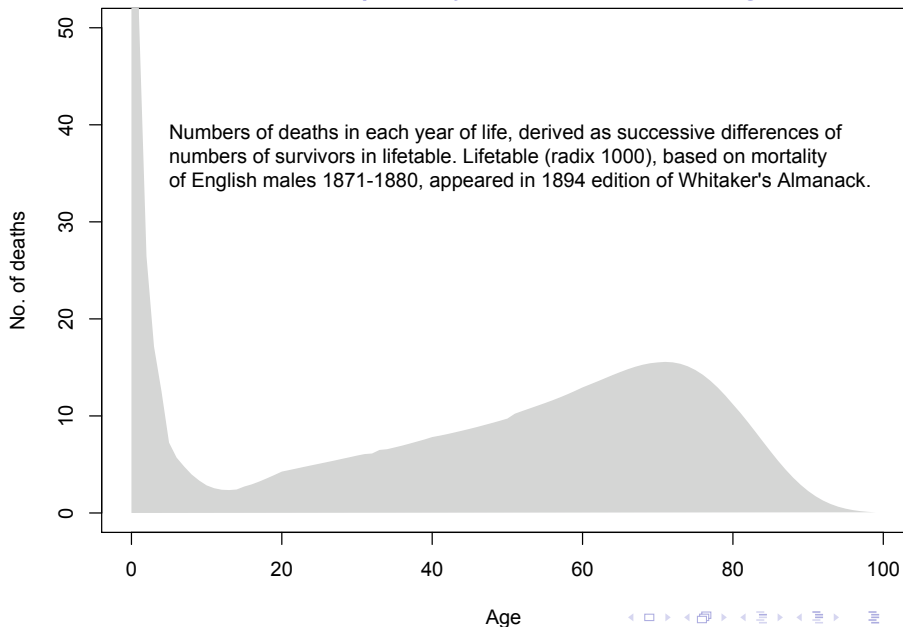


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Idea: Force of mortality external to a person.

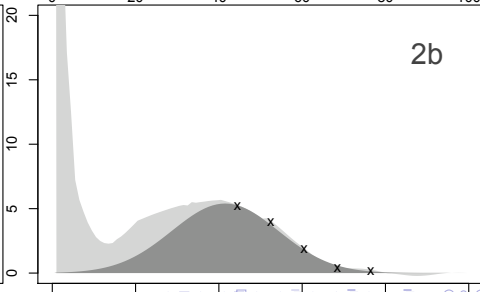
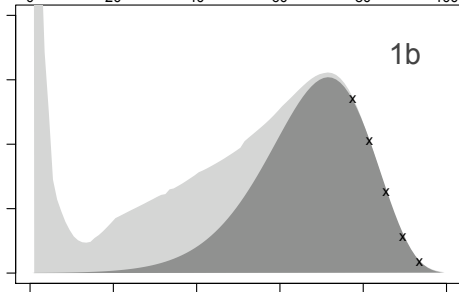
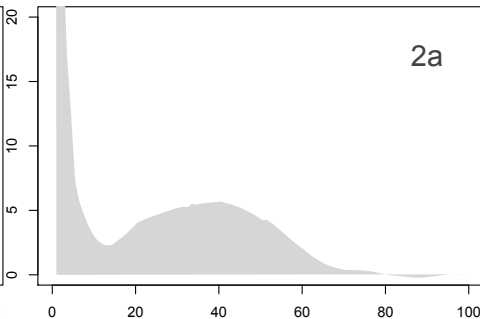
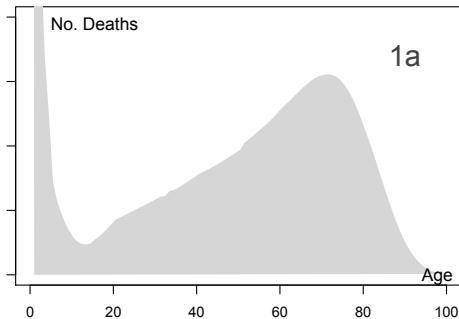
Pearson's data: frequency distribution of age at death

Pearson's data: frequency distribution of age at death



How Pearson fitted the 5-component mixture

How Pearson fitted the 5-component mixture



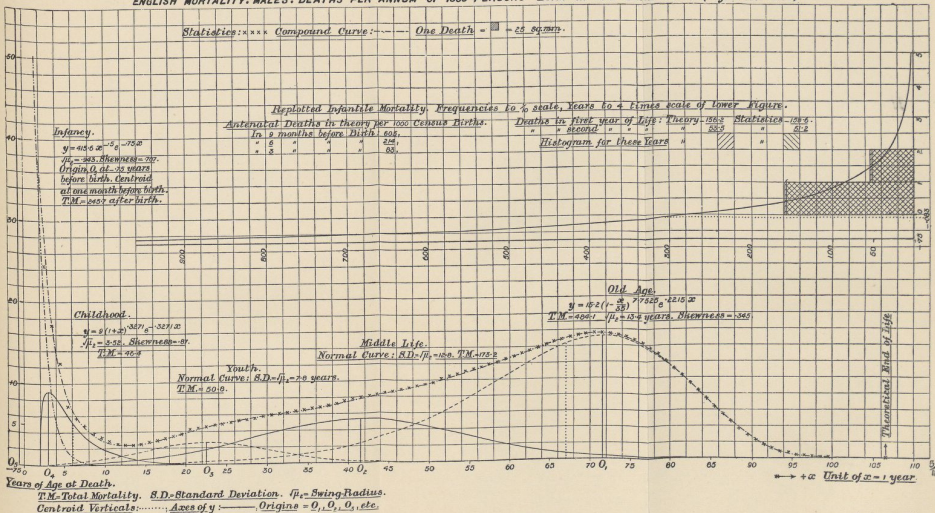
The full 5-component mixture

The full 5-component mixture

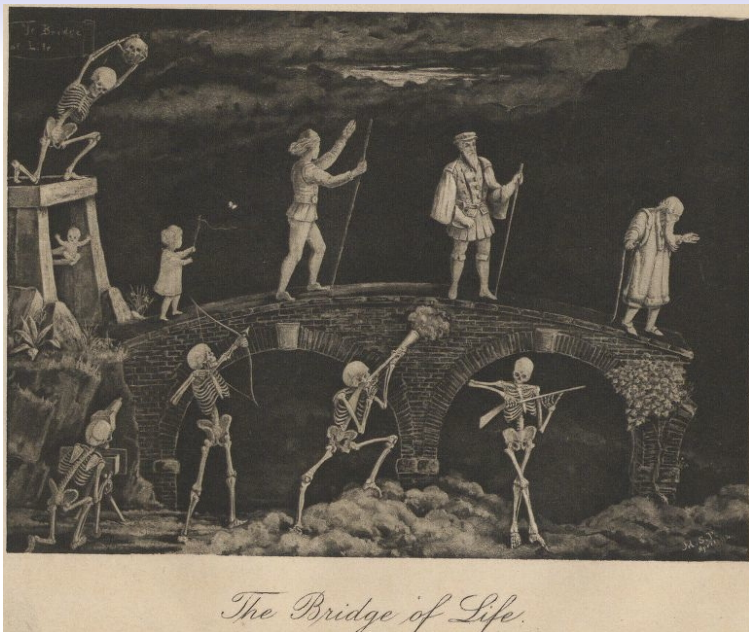
PLATE IV.

ENGLISH MORTALITY. MALES. DEATHS PER ANNUM OF 1000 PERSONS BORN IN THE SAME YEAR. (Ogle: 1871-1880)

Statistics: * * * Compound Curve: - - - One Death = \square = 25 sq. units.



To face page 26.



Rendered by Pearson's wife, Maria Sharpe Pearson.

21st century animation of Pearson's Bridge of Life

21st century animation of Pearson's Bridge of Life

Using hazard (intensity / force of mortality / incidence density)
functions derived from Pearson's 5 curves

<http://www.biostat.mcgill.ca/hanley/BridgeOfLife/>

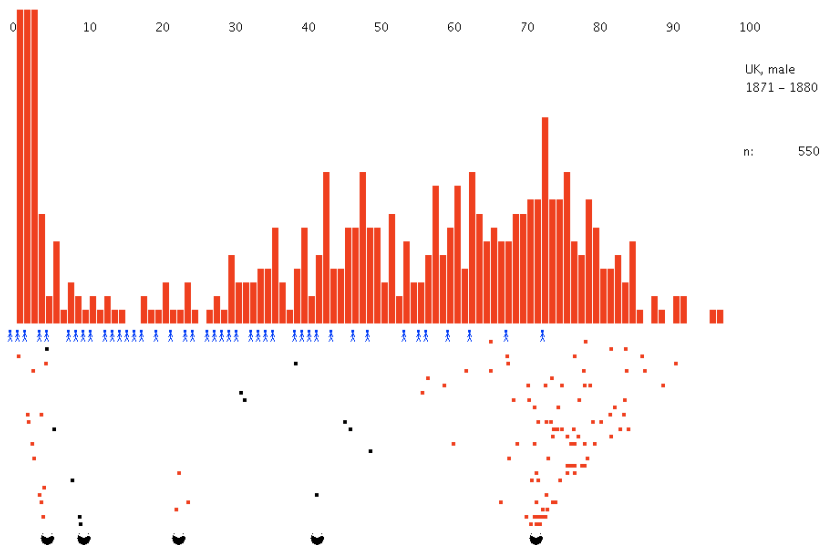
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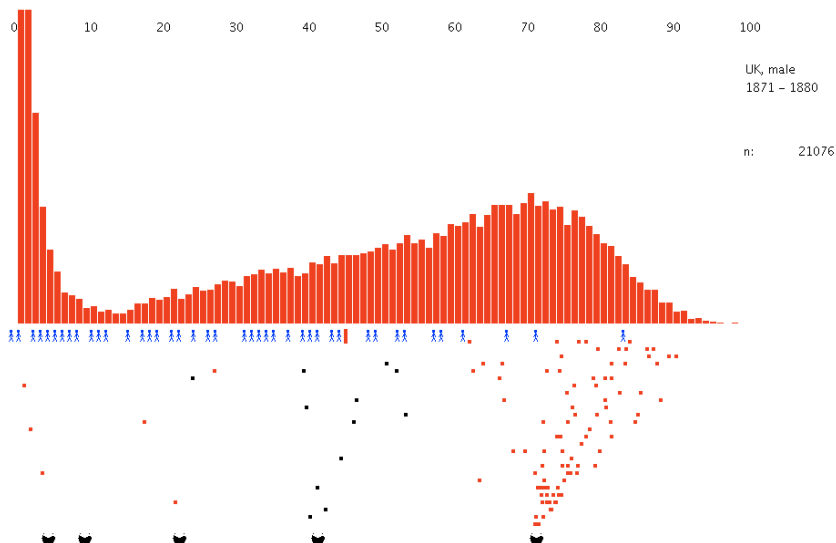
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WARNING: Moving images shown

21st century animation of Pearson's Bridge of Life



21st century animation of Pearson's Bridge of Life



Cultural imagery and statistical models of the force of mortality: Discussion

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- Force of mortality and its components/derived quantities can be difficult to explain

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Teacher	Allegory	Conceptual model	Statistical model
Addison, 1711	Yes	External/Trap-doors	-
Gompertz, 1825	-	Internal/Body wears out	Smooth hazard/survival function
Pearson, 1897	-	External /Marksman	Mixture of frequency functions

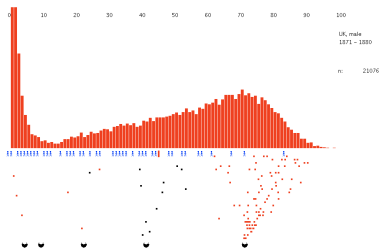
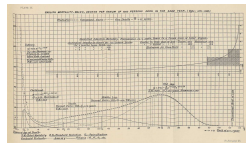
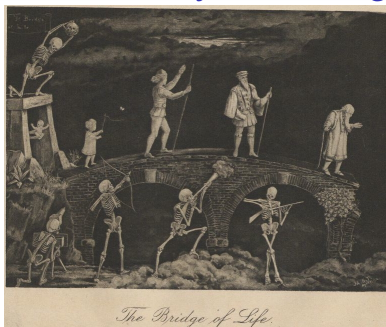
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- Three teachers → 21st century computer animation

Force of Mortality and Bridge of Life



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FUNDING / CO-ORDINATES

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