Note.

On the Probability Distribution of α Particles.

By H. Bateman.

Let λdt be the chance that an α particle hits the screen in a small interval of time dt. If the intervals of time under consideration are small compared with the time period of the radioactive substance, we may assume that λ is independent of t. Now let $W_n(t)$ denote the chance that n α particles hit the screen in an interval of time t, then the chance that $(n+1)$ particles strike the screen in an interval $t+dt$ is the sum of two chances. In the first place, $n+1$ α particles may strike the screen in the interval t and none in the interval dt. The chance that this may occur is $(1-\lambda dt)W_{n+1}(t)$. Secondly, n α particles may strike the screen in the interval t

and one in the interval dt; the chance that this may occur is $\lambda dt W_n(t)$. Hence

$$W_{n+1}(t+dt) = (1-\lambda dt) W_{n+1}(t) + \lambda dt W_n(t).$$

Proceeding to the limit, we have

$$\frac{dW_{n+1}}{dt} = \lambda(W_n - W_{n+1}).$$

Putting $n=0, 1, 2 \ldots$ in succession we have the system of equations:

$$\frac{dW_0}{dt} = -\lambda W_0,$$

$$\frac{dW_1}{dt} = \lambda(W_0 - W_1),$$

$$\frac{dW_2}{dt} = \lambda(W_1 - W_2),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

which are of exactly the same form as those occurring in the theory of radioactive transformations *, except that the time-periods of the transformations would have to be assumed to be all equal.

The equations may be solved by multiplying each of them by $e^{\lambda t}$ and integrating. Since $W_0(0)=1$, $W_n(0)=0$, we have in succession:

$$W_0 = e^{-\lambda t},$$

$$\frac{d}{dt}(W_1 e^{\lambda t}) = \lambda, \qquad \therefore \quad W_1 = \lambda t e^{-\lambda t},$$

$$\frac{d}{dt}(W_2 e^{\lambda t}) = \lambda^2 t, \qquad \therefore \quad W_2 = \frac{(\lambda t)^2}{2!} e^{-\lambda t},$$

and so on. Finally, we get

$$W_n = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

The *average* number of α particles which strike the screen in the interval t is λt. Putting this equal to x, we see that the chance that n α particles strike the screen in this interval is

$$W_n = \frac{x^n}{n!} e^{-x}.$$

* Rutherford, 'Radioactivity,' 2nd edition, p. 330. The chance that an atom suffers n disintegrations in an interval of time t is equal to the ratio of the amount of the nth product present at the end of the interval to the amount of the primary substance present at the commencement.

The particular case in which $n=0$ has been known for some time (Whitworth's 'Choice and Chance,' 4th ed. Prop. 51).

If we use the above analogy with radioactive transformation, the theorem simply tells us that the amount of primary substance remaining after an interval of time t is $e^{-\lambda t}$ if a unit quantity was present at the commencement.

The *probable* number of α particles striking the screen in the given interval is

$$p=\sum_{n=1}^{\infty} n\mathrm{W}_n = xe^{-x}\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}=x.$$

The *most probable* number is obtained by finding the maximum value of W_n.

Since $\frac{\mathrm{W}_n}{\mathrm{W}_{n-1}}=\frac{x}{n}$, this ratio will be greater than 1 so long as $n<x$. Hence if $n \lesseqgtr x$,

$$\mathrm{W}_n \gtreqless \mathrm{W}_{n-1};$$

if $n=x$, $\mathrm{W}_n=\mathrm{W}_{n-1}$. The most probable value of n is therefore the integer next greater than x; if, however, x is an integer, the numbers $x-1$ and x are equally probable, and more probable than all the others.

The value of λ which is calculated by counting the total number of α particles which strike the screen in a large interval of time T, will not generally be the true value of λ. The mean deviation from the true value of λ is calculated by finding the mean deviation of the total number N of α particles observed in time T from the true average number λT. This mean deviation D (mittlerer Fehler) is, according to the definition of Bessel and Gauss, the square root of the probable value of the square of the difference $\mathrm{N}-\lambda\mathrm{T}$, and so is given by the series

$$\mathrm{D}^2=\sum_{n=0}^{\infty}(\mathrm{N}-\lambda\mathrm{T})^2\frac{(\lambda\mathrm{T})^{\mathrm{N}}}{\mathrm{N}!}e^{-\lambda\mathrm{T}}$$

$$=e^{-\lambda\mathrm{T}}\sum_{\mathrm{N}=0}^{\infty}\left[\frac{(\lambda\mathrm{T})^{\mathrm{N}}}{(\mathrm{N}-2)!}+\frac{(\lambda\mathrm{T})^{\mathrm{N}}}{(\mathrm{N}-1)!}-2\frac{(\lambda\mathrm{T})^{\mathrm{N}+1}}{(\mathrm{N}-1)!}+\frac{(\lambda\mathrm{T})^{\mathrm{N}+2}}{(\mathrm{N})!}\right]=\lambda\mathrm{T}.$$

Hence $\mathrm{D}=\sqrt{\lambda\mathrm{T}}$, and the mean deviation from the value of λ is accordingly

$$\frac{D}{T} = \sqrt{\frac{\lambda}{T}};$$

it thus varies inversely as the square root of the length of the interval of time. This result is of the same form as the classical one used by E. v. Schweidler in the paper referred to earlier.

The probable value of $|N - \lambda T|$ (der durchschnittlicher Fehler) is much more difficult to calculate.