Monty's dilemma with no formulas

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Summary The article presents an attempt to analyse Monty's dilemma by means of conversational formula-free dialogues and to simulate the problem by composing isomorphic stories. The crucial roles of specifying the underlying scenarios and explicating epistemic and probabilistic assumptions are highlighted.

Keywords: Teaching; Monty's dilemma; Epistemic assumptions; Probabilistic assumptions; Underlying scenario; Verbal analysis; Isomorphic stories.

INTRODUCTION

In the TV game show 'Let's make a deal', the (supposedly male) contestant is given a choice between three closed doors. Behind one of the doors, chosen at random, there is an attractive prize; behind the other two doors are gag prizes, namely, goats. Suppose the contestant picks door 1. It remains closed and, according to the rules of the game, the host, Monty Hall, who knows what's behind all doors, opens one of the other doors, say door 3, that he knows would reveal a goat. He then gives the contestant the option of moving to door 2 and winning whatever would be found behind the door of his last choice. Should the contestant stick to door 1 or switch to door 2?

A decisive factor for evaluating the probabilities that the prize is behind door 1 or 2 is the underlying detailed scenario, which specifies what everybody knows and what were the probabilistic assumptions and decisions that have resulted in opening door 3 and showing a goat. Numerous writings have discussed and analysed that problem (e.g., Gillman, 1992; Morgan et al., 1991; Nickerson, 1996; Rosenhouse, 2009). They present diverse variations, extensions and possible behind-the-scenes scenarios (Falk, 1992; Eisenhauer, 2000; Puza et al., 2005). I shall focus on only two out of the various scenarios.

Two scenarios

According to the **classic scenario** (e.g., Lucas et al., 2009), the rules of the game dictate which door Monty opens (on the basis of his knowledge of what's behind the doors) if the prize is behind door 2 or 3. And if it is behind door 1 – when he has a choice – he decides *at random*, by flipping a fair coin, whether to open door 2 or 3. Should the contestant stick to door 1, or should he switch to door 2, or does it make no difference? The answer is that the contestant would better switch. His probability of winning the prize by moving to door 2 is 2/3 (see, e.g., Nickerson, 1996, p. 419).

Another underlying situation, the **earthquake scenario**, suggested by MacKay (2003, p. 57), posits that after the contestant has chosen door 1, when the game-show host is about to open one of the other doors, an earthquake rattles the building and one of the three doors flies open. It happens to be door 3, and it shows a goat. It is postulated that all three doors are as likely to be opened, irrespective of whether or not the door has initially been picked by the contestant or whether it does or doesn't conceal the prize. Should the contestant now stick to door 1, or switch to door 2, or does it make no difference? The answer is that doors 1 and 2 remain equally likely to hide the prize. There is no reason for the contestant to prefer either door (detailed analysis by MacKay, p. 61).

Using words only

Some students in secondary school and introductory college courses are averse to formulas and prefer avoiding them. Although this tendency shouldn't be encouraged, the challenge of coping with this celebrated teaser by formula-free verbal means is worth a trial. The relevant literature (e.g., Eisenhauer, 2000; Lucas et al., 2009; Puza et al., 2005) relies heavily on two-dimensional tables, tree diagrams and expressions loaded with symbols. In what follows, I attempt analysing Monty's problem by ordinary prose, using verbal exchanges. In addition, I propose modelling the problem by delineating equivalent semi-realistic stories.

CONVERSATIONS ABOUT MONTY'S

Inspired by McEwan's (2013) novel (pp. 236–238), let's imagine dialogues between a mathematician (Math) and a layman (Lay) about Monty's dilemma, with reference to the above two underlying scenarios.

Classic scenario

Lay. It's obvious. With door 1 I had a one in three chance to start with. When door 3 is opened, my options narrow to one in two. And what is true for door 1 is true for door 2. There are therefore equal chances that my prize is behind either of the unopened doors. It makes no difference whether I move or not.
Math. You'd be in good company with that argument. But you'd be wrong! If you go for the other door you double your chances of winning the prize: Your chances that door 1 hides the prize remain one in three. Recall that if the prize is behind door 1, the probability that Monty opens door 3 is one half, and if it is behind door 2, it is certain that he opens door 3. Hence, since doors 1 and 2 are equally probable to start with, it stands to reason that the opening of door 3 renders door 2 double as likely to hide the prize as door 1. So the probability is two thirds that the prize is behind door 2.
Lay. But this Monty has chosen *at random* the door behind which to hide the prize. There are now only two doors where it can be, so randomness implies equal chances, it's one or the other.
Math. That would be true if you had come in the room when door 3 has already been opened and then you'd be told that Monty decided at random where to put the prize and you'd be asked to choose between doors 1 and 2. Then you'd be looking at odds of one in two.
Lay (confused). But Monty has given me new information, after my picking door 1, by opening door 3. My chances were one in three. Now there are only two options, therefore the chances became one in two.
Math. Look at it that way: Monty opened door 3 either as a result of a toss-up between doors 2 and 3 when the prize is behind door 1, or because he knew that the prize is behind door 2. You got no new information concerning door 1, its probability of hiding the prize remains one third. Door 2, in contrast, has passed a certain test: it could have been opened and it hadn't, maybe because the prize is there; this augments its prior chances of hiding the prize above one third, so you'd better move there.

It is important to realize that having two options doesn't always mean they are equiprobable. Not all uncertain situations guarantee equal probabilities to all available alternatives.
Lay. I see what you mean. Thank you. I'll keep thinking about it.

Earthquake scenario

Lay. You have propounded a bizarre situation. Nevertheless, I fail to see the difference between the two set-ups: The scenes look exactly alike. In both cases the contestant chooses door 1 and door 3 is open revealing a goat. Your previous rationale implies that one should now switch to door 2.
Math. Alas, probability problems are devious: The displays are indeed identical in both cases, but, given the different behind-the-scenes particulars, one has to delve beneath the surface and consider the history of the current scene. Unlike the case of the classic scenario – where the host knows what is behind each door, what the contestant picked, and is committed to avoid the contestant's choice, to open a door with a goat, and to decide at random when there is a choice – the seismic force of the present scenario is completely *blind* and devoid of any responsibility. Each of the three doors is equally likely to be opened by it, the contestant's initial choice and the door hiding the prize notwithstanding. Doors 1 and 2 were a priori equally likely to hide the prize. They remain so after the fortuitous opening of door 3 by the quake, which, in contrast to the situation in the classic case, was not a result of any intentional decision or constraint. One should justifiably feel indifferent when confronted with a choice between these two doors.

Conclusion

Lay. Would you agree to my moral from these two scenarios: One has to carefully consider the background story and the assumptions behind the problem's givens: What was the state of knowledge of the involved participants? Was seeing a goat behind door 3 an outcome of some random event (of what probability)? Or was it a result of some human probabilistic decision?
Math. Bravo, just right!

ISOMORPHIC STORIES

I was further inspired by McEwan's (2013, pp. 240–246) attempts of representing Monty's story via an analogous real-life story to try some other stories that would be structurally equivalent to the above two scenarios and, unlike McEwan's case, would not repeat the situation of three closed doors. The purpose is to represent the probabilistic assumptions, the epistemic situation (who knows what) and the problems' givens by corresponding realistic background features and events. The same solutions, as specified above, would then apply to appropriate cover stories, irrespective of their particular contents. My attempts are presented below as instances of such an exercise. After analysing the problem verbally in class, students might be challenged by the assignment of devising isomorphic stories.

The task of simulating the problem in a new context requires consideration of all the antecedent conditions and assumptions. One has to map every constituent of the puzzle to a corresponding story component. A satisfactory accomplishment of that task would indicate in-depth understanding of the puzzle.

Background

A hit-and-run accident occurred at the outskirts of a city. The (single) guilty driver couldn't be identified, but the description of the hitting car led the police to the Smith family, on whose car there were telltale marks that established that car undoubtedly as the one involved in the accident. The only equally suspected drivers were the three adult sons of the family: Tom, Dick and Harry.

One-to-one correspondence of features

Table 1 presents features of Monty's dilemma paired one-to-one with those of the attempted hit-and-run stories, so as to render the structure of the two systems mathematically parallel.

Classic scenario

The police summoned Tom (at random) for interrogation. However, when Tom was on the verge of getting to the station, Harry arrived and showed a snapshot of himself, taken by his girlfriend at exactly the point of time of the accident. The location of that photo was identified as being far away from the scene of the crime. This provides an ironclad alibi for Harry. It stands to reason that an innocent man would be able to provide an alibi and that were both Dick and Harry innocent, it would be only a question of equal chances who'd come up first with exonerating evidence.

The problem now is, should the police focus on investigating Tom or move to Dick, for whom there is no evidence of his whereabouts at the critical time? Who of the two is more likely to be the culprit, or are they equally likely?

Earthquake scenario

The police summoned Tom for interrogation. However, when Tom was on the point of arriving at the station, a young rash policeman jumped the gun and recklessly decided to investigate the matter on his own. He totally ignored previous steps taken by the police and chose one suspect out of the three for interrogation *at random*. It happened to be Harry. The latter provided a newspaper photograph showing a trendy wedding party, held at the exact time of the accident, but miles away. Harry was clearly seen there rising his glass to the bride and bridegroom. This cast-

Table 1. One-to-one correspondence between elements of Monty's and those of the realistic stories

Monty's game show	Hit-and-run stories
Doors 1, 2 and 3	Suspects Tom, Dick and Harry
The door that hides the prize	The guilty suspect
Contestant chooses a door	Police summons a suspect for interrogation
A goat is revealed behind a door	A suspect provides an airtight alibi and is exonerated
Monty knows what's behind each door and doesn't divulge it to the TV audience	The author knows who is guilty and who is innocent and doesn't divulge it to the readers
Monty opens a door, other than the contestant's choice, showing a goat. If the two doors not chosen by the contestant hide goats, he decides at random which door to open	A suspect, other than the one summoned by the police, reveals an alibi. If the two suspects not summoned by the police are innocent, it is a toss-up who'd be the one to reveal an alibi
An earthquake opens one random door out of the three	A young impulsive policeman rushes to interrogate one random suspect out of the three

iron alibi clearly exculpates Harry. Only Tom and Dick are now suspects. Should the police stick to Tom or switch to investigate Dick? Who of the two is more likely to be the culprit, or are they equally likely?

DISCUSSION

Undoubtedly, verbal discussions in social and educational contexts can advance people's apprehension of Monty's dilemma. However, an attempt to reproduce the same in a different guise might bring about a much more thorough penetration of the problem.

My hunch is that young students would welcome this kind of challenge, and they might do better than me in inventing interesting, attractive stories. They are initially bound to be wrong in places – one-to-one correspondences might be inaccurate, and assumptions could be disregarded or misapplied – but then corrective procedures would be most instructive, as, in fact, happened in the process of producing my stories (as well as in McEwan, 2013, pp. 240–246).

Composing stories isomorphic to classic Monty and the earthquake variety must be based on full awareness of the details of behind-the-scenes procedures and the underlying assumptions. Epistemic assumptions concerning the state of knowledge of all participants are no less important than the probabilistic ones. All this has to be mapped into appropriate ingredients of the down-to-earth stories. Success in that undertaking may serve as a testament to having mastered Monty's dilemma. And reliance on isomorphism, based on one-to-one correspondence between the elements of two systems, would undoubtedly be educationally valuable. Hopefully, the present endeavour might motivate some readers to contribute more stories isomorphic to Monty's or to other famous problems.

This experience may pave the way to the next step of introducing the symbols for the various events and the conditional probabilities that specify the relations between the events. For example, if M_3 denotes the event that Monty opens door 3 showing a goat, and D_1 and D_2 designate his knowing that the prize is behind door 1 and door 2, respectively, then – after having coped with the dilemma verbally and/or via a story – $P(M_3|D_2)=1$ and $P(M_3|D_1)=1/2$ may be comprehended easily. This would promote students' understanding and use of the mathematical language, which is more succinct and potent than the spoken language.

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