MISCELLANEA.

(1) Fiducial Limits for the Poisson Distribution.

By F. GARWOOD, Ph.D.

1. *The Method of Fiducial or Confidence Limits.*

A situation of very common occurrence in statistics arises when a random sample is drawn from a population which is not completely specified, and it is desired to draw some inference about the population on the basis of the sample. It is usual to assume, from independent evidence, a mathematical form for the chance distribution of the variate, so that it will be completely specified if the value of one or more parameters is known. Thus past experience may tell us that the population of heights of men belonging to a homogeneous race group is represented by the familiar normal distribution, of which the parameters necessary to specify it are the mean and standard deviation. Any inference which can be made about the parameters on the basis of the sample is of course subject to error, since two different populations can give rise to the same sample. It is the fundamental property of the method of fiducial limits*, however, that the risk of making incorrect inferences by this method can be controlled. The value of the method is therefore that if a large number of situations arise in which the rules of the method are applied, one can have confidence that, in the long run, the proportion of inferences which are false does not exceed appreciably a pre-determined ratio, say 5 °/₀, or, if more stringency be required, 1 °/₀ or ·01 °/₀.

2. *Application of the Method to the Poisson Distribution*†.

The Poisson limit to the binomial defines a hypothetical population containing all the positive integers, including zero, and the proportion of times the integer x occurs is

$$\frac{e^{-m}m^x}{x!} \qquad \text{...(1)},$$

where m is the mean, the only parameter necessary to define the distribution.

Suppose we are confronted by a physical phenomenon which is known to give rise to a Poisson distribution, such as the occurrence of random and independent events in time or space, and we wish to infer something about the mean on the data of one observation, which is an integer. Suppose further that we do not wish the risk of error to rise above a probability of ·05, and that we are only interested in inferring a lower limit above which the mean lies. The rule to be followed is that we look up the entry in the 5 °/₀ "lower limit" column of the Table of fiducial limits, given below, corresponding to the sample integer observed (or interpolate into the table if necessary), and make the statement that the mean lies above this. As stated above, this rule, which may be applied in sampling from a series of different distributions, will lead to false statements being made on a proportion of occasions which is never much more than once in 20 in the long run, though of course one cannot tell on *which* particular occasions the rule has failed.

* R. A. Fisher, *Proc. Camb. Phil. Soc.* xxvi. (1930), p. 528; *Proc. Roy. Soc.* A, cxxxix. (1933), p. 343. See also J. Neyman, *Journ. R. Statistical Soc.* xcvii. (1934), p. 589. Neyman uses the term "confidence limit."

† The reader is referred to a paper entitled "Statistical principles of routine work in testing clover seeds for dodder" by J. Przyborowski and H. Wileński, *Biometrika*, xxvii. (1935), pp. 273—292, dealing with somewhat the same problem. [Dr Garwood's paper was part of his thesis for the Ph.D. degree of London University completed in June 1934. Its publication has unfortunately been delayed overlong. Ed.]

Similar results hold for the 1 °/。 lower limit, for the upper limit, and for the upper and lower limits combined; in the last cases, the probabilities of error are, respectively, not greater than ·10 and ·02.

This is an example of the application of the method of fiducial limits to a distribution in which the variate can only assume discrete values; it is important to note that in such cases the table of fiducial limits can only be calculated so that the probability of error will not exceed certain values, whereas in the case of continuous distributions it can be fixed exactly at a desired set of values. The consideration of the actual value of the probability of error is returned to later.

3. *Theory of the Method.*

From equation (1) it follows that the probability of the occurrence of either 0, 1, 2, ... or x as one random sample from the Poisson distribution with mean x is

$$P\{x \mid m\} = e^{-m}\left(1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots\ldots + \frac{m^x}{x!}\right) \quad \ldots\ldots (2).$$

By differentiating this with respect to m, it is seen that

$$P\{x \mid m\} = \int_m^\infty \frac{e^{-t}t^x}{x!}\,dt \quad \ldots\ldots (3),$$

$$= 1 - \int_0^m \frac{e^{-t}t^x}{x!}\,dt \quad \ldots\ldots (4).$$

Corresponding to values of x from 0 to 50 we have calculated the values of m for which

$$P\{x-1 \mid m\} = \cdot 99 \text{ and } \cdot 95 \quad \ldots\ldots (5),$$

and

$$P\{x \mid m\} = \cdot 05 \text{ and } \cdot 01 \quad \ldots\ldots (6).$$

These are the fiducial limits for m corresponding to x, which we have denoted by $m_{\cdot 99}(x)$, $m_{\cdot 95}(x)$, $m_{\cdot 05}(x)$, and $m_{\cdot 01}(x)$ respectively, and they are given in Table I*. The lower limits corresponding to $x = 0$ are defined as zero.

To demonstrate the fundamental property of the fiducial limits, consider a Poisson distribution with mean $m = 15$, say, and the upper 1 °/。 limit for m corresponding to samples drawn from it. There is an integer s such that

$$m_{\cdot 01}(s) \leqq m = 15 < m_{\cdot 01}(s+1) \quad \ldots\ldots (7),$$

and from the column for $m_{\cdot 01}(x)$ we see that $s = 6$. If now the statements "$m < m_{\cdot 01}(x)$" are made according as the various values of x arise in random sampling, a false statement will be made when x is either 0, 1, 2, 3, 4, 5 or 6 $(=s)$, since the statement will then be one of the following: "$m < 4\cdot 61$," "$m < 6\cdot 64$," ...... "$m < 14\cdot 57$," all of which are false in this case. The probability of one of these events occurring is, by definition,

$$P\{s \mid m\} = P\{6 \mid 15\};$$

this function decreases with increasing m, and since $m \geqq m_{\cdot 01}(s)$, we must have

$$P\{6 \mid 15\} = P\{s \mid m\} \leqq P\{s \mid m_{\cdot 01}(s)\} = \cdot 01 \quad \ldots\ldots (8),$$

by definition of $m_{\cdot 01}(s)$, (see equation (6)). It follows that if the above rule is observed in sampling from any Poisson distribution, i.e. if the statement

$$m < m_{\cdot 01}(x) \quad \ldots\ldots (9)$$

is made according to whatever integer x arises, then the probability of a false statement is

$$P\{s \mid m\} \equiv P(m) \quad \ldots\ldots (10),$$

* The upper limits correspond to those given by Przyborowski and Wileński in their Table V (*loc. cit.* p. 288). These authors, however, only give the limit to 1 place of decimals and have not tabled any lower limits.

where s is given by

$$m_{\cdot 01}(s) \leq m < m_{\cdot 01}(s+1) \quad \text{.......(11)},$$

and this probability does not exceed ·01. Clearly, if the populations sampled have means less than 4·61, the statement (9) will never be false.

TABLE I.

Fiducial Limits for Mean of Poisson Distribution.

Observed Number x	Lower Limits		Upper Limits	
	$m_{\cdot 99}(x)$	$m_{\cdot 95}(x)$	$m_{\cdot 05}(x)$	$m_{\cdot 01}(x)$
0	0·0000	0·0000	3·00	4·61
1	0·0101	0·0513	4·74	6·64
2	0·149	0·355	6·30	8·41
3	0·436	0·818	7·75	10·05
4	0·823	1·37	9·15	11·60
5	1·28	1·97	10·51	13·11
6	1·79	2·61	11·84	14·57
7	2·33	3·29	13·15	16·00
8	2·91	3·98	14·43	17·40
9	3·51	4·70	15·71	18·78
10	4·13	5·43	16·96	20·14
11	4·77	6·17	18·21	21·49
12	5·43	6·92	19·44	22·82
13	6·10	7·69	20·67	24·14
14	6·78	8·46	21·89	25·45
15	7·48	9·25	23·10	26·74
16	8·18	10·04	24·30	28·03
17	8·89	10·83	25·50	29·31
18	9·62	11·63	26·69	30·58
19	10·35	12·44	27·88	31·85
20	11·08	13·25	29·06	33·10
25	14·85	17·38	34·92	39·31
30	18·74	21·59	40·69	45·40
35	22·72	25·87	46·40	51·41
40	26·77	30·20	52·07	57·35
45	30·88	34·56	57·69	63·23
50	35·03	38·96	63·29	69·07

Similar reasoning applies to the lower 1 °/。 limit, and it can be shown that if one follows the rule of making the statement

$$m \geq m_{\cdot 99}(x) \quad \text{.......(12)}$$

about the mean according to whatever value of x may arise in sampling, then the probability of a false statement is

$$P'(m) \quad \text{.......(13)},$$

which does not exceed ·01. By combining the two limits the risk of error involved in making the statement

$$m_{\cdot 99}(x) \leq m < m_{\cdot 01}(x) \quad \text{.......(14)}$$

is

$$P''(m) \equiv P(m) + P'(m) \quad \text{.......(15)},$$

which does not exceed ·02.

If the 5 °/。 limits are used, the range in which m may be predicted to lie is narrowed, but at the expense of increasing the possible risk of error from ·02 to ·10.

The values of the functions $P(m)$, $P'(m)$, and of their sum $P''(m)$, are shown in Figs. 1 and 2 for two ranges of m. $P(m)$ and $P'(m)$ are discontinuous at those values of m which are respectively upper and lower 1 °/₀ fiducial limits (i.e. the values of m given under $m_{\cdot 01}(x)$ and $m_{\cdot 99}(x)$ of Table I), and the functions reach up to ·01 at these points. $P''(m)$ is discontinuous at both these sets of points, but, over the range investigated, it is always less than ·02. Up to $m = 4{\cdot}61$, $P(m)$ is zero, and hence $P''(m) = P'(m)$. Fig. 1 gives the graph of $P'(m)$ for a small range of m, starting from zero, and shows the first three branches of the function. In Fig. 2 the three functions are plotted from $m = 4{\cdot}0$ to 10·6 on a larger horizontal scale.

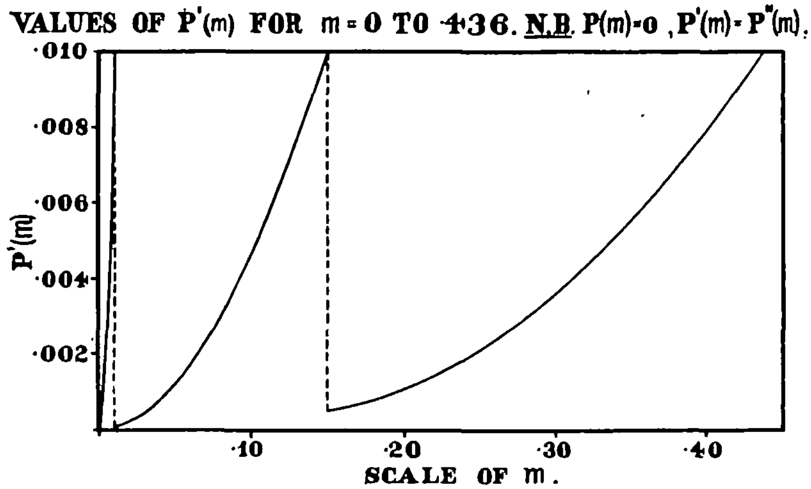

Fig. 1.

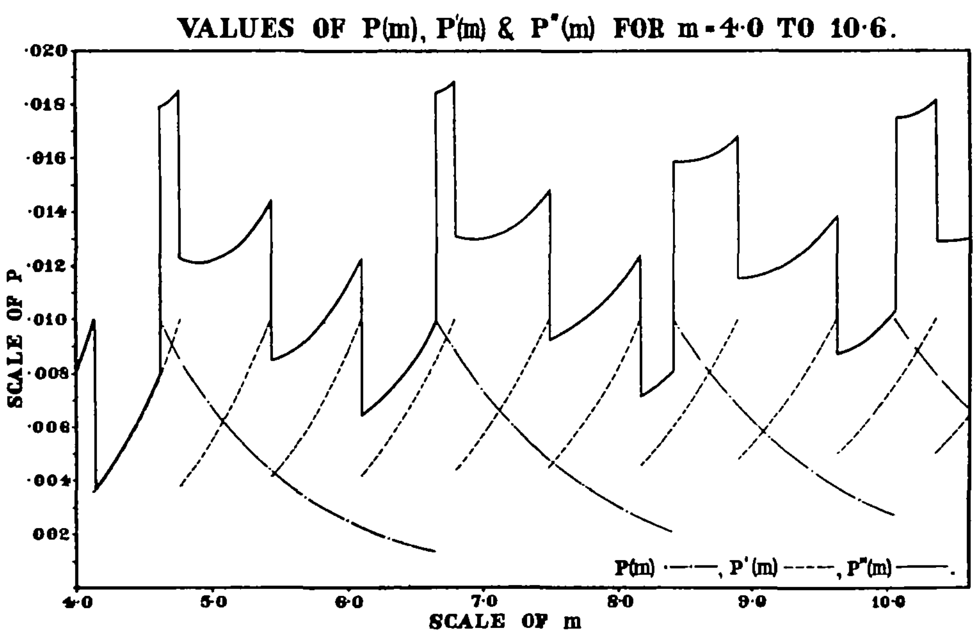

Fig. 2.

In the application of the fiducial method, the values of m are never known, so that the values recorded in the figure cannot be used exactly. If an *a priori* distribution of means could be assumed, then by integration of $P(m)$, $P'(m)$ and $P''(m)$ one could calculate the absolute risk of error in the fiducial statements, and the result would be less than ·01, ·02, ·05, ·10, as the case may be. This situation rarely arises in practice, but Fig. 2 shows that the limits given to the probability of error are very conservative ones and the true probability may often be considerably less.

4. *Method of Calculation of Table I.*

For values of x up to 14, the lower limits were calculated from Fisher's χ^2 Table* by means of the transformations:

$$x=\frac{n-2}{2},\quad m=\tfrac{1}{2}\chi^2,\quad (P=\cdot 99 \text{ and } \cdot 95) \qquad\qquad (16).$$

For values of x up to 15, the upper limits were calculated from the same table with

$$x=\frac{n}{2},\quad m=\tfrac{1}{2}\chi^2,\quad (P=\cdot 01 \text{ and } \cdot 05) \qquad\qquad (17).$$

This can be done because the distribution of χ^2, with n degrees of freedom, has a probability integral, given in the table, equal to

$$P=\int_{\frac{1}{2}\chi^2}^{\infty} \frac{e^{-\frac{\chi^2}{2}}\left(\frac{\chi^2}{2}\right)^{\frac{n-2}{2}}}{\frac{n-2}{2}!}\, d\left(\frac{\chi^2}{2}\right) \qquad\qquad (18)$$

which reduces to (3) after the necessary transformations have been applied.

For the remainder of the table, inverse interpolation was made into the *Tables of the Incomplete* Γ*-Function*†, which is a table of the function

$$I(u,p)=\int_0^{u\sqrt{p+1}} \frac{e^{-t}\,t^p}{p!}\,dt = 1-P(p\,|\,u\sqrt{p+1}) \qquad\qquad (19).$$

5. *Approximate Formulae for Large Samples.*

The Γ-Function Table stops at $p=50$, so that approximations must be used to calculate the fiducial limits for m corresponding to values of x greater than 50. At the foot of his χ^2 Table, Fisher suggests that for $n>30$, $\sqrt{2\chi^2}-\sqrt{2n-1}$ may be used as a normal variate with unit standard deviation. A more accurate formula, though less simple, has been given by Wilson and Hilferty‡. This assumes that $\left(\frac{\chi^2}{n}\right)^{\frac{1}{3}}$ is normally distributed about $1-\frac{2}{9n}$ with standard deviation equal to $\sqrt{\frac{2}{9n}}$. Fiducial limits for $m=\frac{1}{2}\chi^2$ have been calculated from these approximations for $x=20$, 30, 40, 50, and compared in Table II with the true values obtained from the Γ-Function Table. The error in the limits based on Fisher's approximation only appears to decrease with increasing x in one instance, whereas the error in Wilson's and Hilferty's formula decreases in all cases and is much smaller.

By using the equations (16) and (17) this table also serves as a comparison of the approximations for the significance levels of χ^2 for large values of n, the number of degrees of freedom.

* R. A. Fisher, *Statistical Methods for Research Workers*, Table III.

† Edited by Karl Pearson (1922). ‡ *Nat. Acad. Sci.* XVII. No. 12 (1931), p. 684.

TABLE II.

Comparison of Approximate Formulae for Fiducial Limits of m for Large Values of x.

m_T = True value, obtained from Γ-Function Tables.
m_F = Approximate value, obtained from Fisher's formula.
m_W = Approximate value, obtained from Wilson's and Hilferty's formula.

		x	m_T	m_F	$m_T - m_F$	m_W	$m_T - m_W$
Lower Limits	1 %	20	11·082	10·764	·318	11·070	·012
		30	18·742	18·414	·328	18·732	·010
		40	26·770	26·436	·334	26·761	·009
		50	35·032	34·694	·338	35·025	·007
	5 %	20	13·255	13·116	·139	13·254	·001
		30	21·594	21·455	·139	21·594	·000
		40	30·196	30·056	·140	30·196	·000
		50	38·965	38·825	·140	38·965	·000
Upper Limits	1 %	20	33·103	32·700	·403	33·113	−·010
		30	45·401	45·003	·398	45·409	−·008
		40	57·347	56·953	·394	57·355	−·008
		50	69·067	68·676	·391	69·074	−·007
	5 %	20	29·062	28·919	·143	29·060	·002
		30	40·691	40·548	·143	40·689	·002
		40	52·069	51·926	·143	52·068	·001
		50	63·287	63·144	·143	63·286	·001

(ii) Note on Karl Pearson's Paper: "On a method of ascertaining limits to the actual number of marked members in a population of given size from a sample." [*Biometrika*, Vol. xx^A. pp. 149–174.]

By K. RAGHAVAN NAIR, M.A.

Madras University.

On page 151 of this paper Professor Karl Pearson has proved at some length the following result:

$$1+\frac{r+1}{1}\frac{N-n}{N-r}+\frac{(r+1)(r+2)}{1.2}\frac{(N-n)(N-n-1)}{(N-r)(N-r-1)}+\dots+\frac{|\underline{s}}{|\underline{r}}\frac{|\underline{N-s}}{|\underline{N-r}}=\frac{|\underline{N+1}}{|\underline{n+1}}\frac{|\underline{n-r}}{|\underline{N-r}}\quad\dots\dots(1),$$

where $r+s=n$ and all the letters denote positive integers.

The problem discussed in this paper is the frequency distribution of populations of size N with number of marked individuals varying from r, $r+1$, ... to $N-s$, obtained on proceeding by the method of Inverse Probability from the knowledge that a sample of n contains r marked and s unmarked individuals. The terms on the left-hand side of (1) are respectively proportional to the probabilities of populations of size N containing r, $r+1$, $r+2$, ... $N-s$ marked individuals. It is clear, therefore, that there are $N-r-s+1$ or $N-n+1$ terms in the above series.