Lecture notes on likelihood function

Statistical Modeling

- Results of a statistical analysis have to be generalizable to be scientifically useful
- A model proposes a general functional relation between the unknown parameter(s) and the observed data. It allows us to make predictions.
- The goal of a statistical analysis is to estimate the unknown parameter(s) in the proposed model
- The likelihood function is central to the process of estimating the unknown parameters. Older and less sophisticated methods include the method of moments, and the method of minimum chi-square for count data. These estimators are not always efficient, and their sampling distributions are often mathematically intractable.

Example: Least squares vs. linear regression: One can fit a best straight line to explain the relation between two continuous variables X and Y using a least squares approach. The intercept (β_0) and slope (β_1) parameters can be estimated as the values that minimize the squared loss: $\sum(Y_i - \beta_0 - \beta_1 X_i)^2$. Using this approach, we cannot say anything about the precision of our estimates, or about future values of Y for a given X. In order to do so, we would need to make some assumption about the probability distribution of the error term, $\epsilon_i = Y_i - \beta_0 - \beta_1 X_i$.

Example: Binned data: Likelihood provides a very flexible approach to combining data, provided one has a probability model for them. As a simple example, consider the challenge of estimating the mean μ from several independent observations for a $N(\mu, \sigma)$ process, but where each observation is recorded to a different degree of numerical 'rounding' or 'binning.' For example, imagine that because of the differences with which the data were recorded, the $n = 4$ observations are $y_1 \in [4, 6),\ y_2 \in [3, 4),\ y_3 \in [5, \infty),\ y_4 \in [-\infty, 3.6)$. Even if we were told the true value of σ, the least squares method cannot handle this uni-parameter estimation task. Exercise: Using the graphical methods described below determine the most plausible value of μ.

Likelihood

Let $X_1, X_2, ..., X_n$ have a joint density function $f(X_1, X_2, ..., X_n|\theta)$. Given $X_1 = x_1, X_2 = x_2, ..., X_n = x_n$ is observed, the function of θ defined by:

$$L(\theta) = L(\theta|x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n|\theta) \tag{1}$$

is the *likelihood function.*

- The likelihood function is not a probability density function.
- It is an important component of both frequentist and Bayesian analyses
- It measures the support provided by the data for each possible value of the parameter. If we compare the likelihood function at two parameter points and find that $L(\theta_1|x) > L(\theta_2|x)$ then the sample we actually observed is more likely to have occurred if $\theta = \theta_1$ than if $\theta = \theta_2$. This can be interpreted as θ_1 is a more plausible value for θ than θ_2.

Example: American or Canadian M&M's? (Discrete parameter): M&M's sold in the United States have 50% red candies compared to 30% in those sold in Canada. In an experimental study, a sample of 5 candies were drawn from an unlabelled bag and 2 red candies were observed. Is it more plausible that this bag was from the United States or from Canada? The likelihood function is: $L(p|x) \propto p^2(1-p)^3$, p=0.3 or 0.5. $L(0.3|x) = 0.03087 < 0.03125 = L(0.5|x)$, suggesting that it is more plausible that the bag used in the experiment was from the United States.

Likelihood Principle

If x and y are two sample points such that $L(\theta|x) \propto L(\theta|y) \; \forall \; \theta$ then the conclusions drawn from x and y should be identical.

Thus the likelihood principle implies that likelihood function can be used to compare the plausibility of various parameter values. For example, if $L(\theta_2|x) = 2L(\theta_1|x)$ and $L(\theta|x) \propto L(\theta|y) \; \forall \; \theta$, then $L(\theta_2|y) = 2L(\theta_1|y)$. Therefore, whether we observed x or y we would come to the conclusion that θ_2 is twice as plausible as θ_1.

Example: Two samples, same distribution: Consider the distribution $Multinomial(n = 6, \theta, \theta, 1-2\theta)$. The following two samples drawn from this distribution have the same likelihood:

$$X = (1,3,2) \Rightarrow \frac{6!}{1!3!2!}\theta^1\theta^3(1-2\theta)^2$$
$$X = (2,2,2) \Rightarrow \frac{6!}{2!2!2!}\theta^2\theta^2(1-2\theta)^2$$

This means both samples would lead us to the same conclusion regarding the relative plausibility of different values of θ.

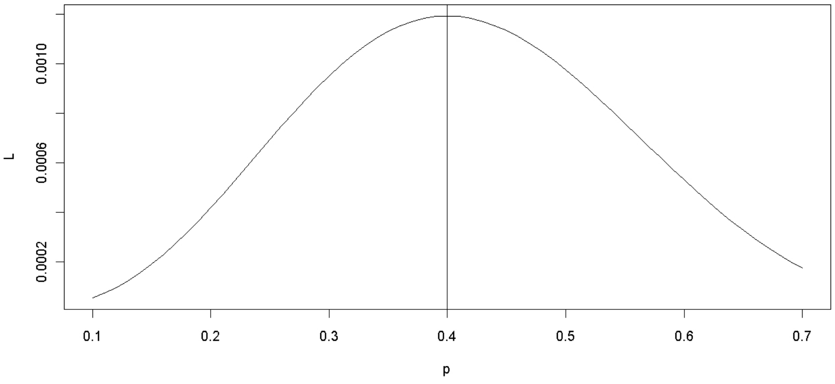
Example: Same sample, two distributions: Two treatments A and B were compared in each of 6 patients. Treatment A was better in the first 5 patients, and treatment B in the sixth patient. Let us imagine this experiment was conducted by two investigators, each of whom, unbeknownst to the other, had a different study design in mind. The first investigator had originally planned to study 6 patients (a binomial experiment), while the second had planned to stop the study when the first patient who preferred treatment B was observed (a negative binomial experiment). Thus, the likelihood function according to the first investigator would be $L(p|x) = \binom{6}{1} p^5(1-p)$, where p=P(Treatment A is preferred). The likelihood function according to the second investigator would be $L(p|y) = \binom{5}{0}p^5(1-p)$. (Note: The negative binomial density function for observing y failures before the r^{th} success is $P(Y = y) = \binom{y+r-1}{r-1}p^r(1-p)^k, k = 0,1,2,3,...)$. The likelihood functions of both investigators are proportional implying that they should reach the same conclusion.

If were to use a frequentist approach for inference and calculate a p-value for the null hypothesis $H_0 : p = 0.5$ against the alternative $H_1 : p > 0.5$, then the first investigator would obtain a p-value of 0.11, while the second investigator would obtain a p-value of 0.03, potentially leading them to different conclusions. (See Goodman S, Towards evidence-based medical statistics. 1: The p-value fallacy. Annals of Internal Medicine, 1999)

Plotting the likelihood function

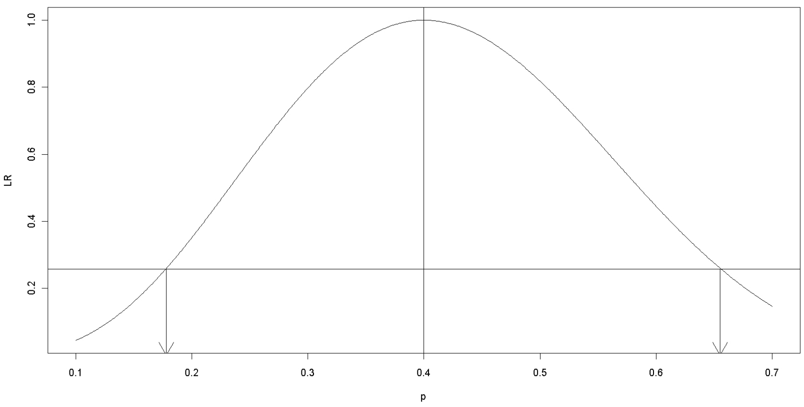
Example from Clayton and Hills (Continuous parameter): In section 3.1, Clayton and Hills describe a hypothetical study in which 10 patients are followed for a fixed time period at the end of which 4 of them die. The likelihood function is given by: $L(p|x) \propto p^4(1-p)^6$. The likelihood of p=0.5 is 9.77×10^{-4}, whereas the likelihood of p=0.1 is 5.31×10^{-5}.

Likelihood function plot:

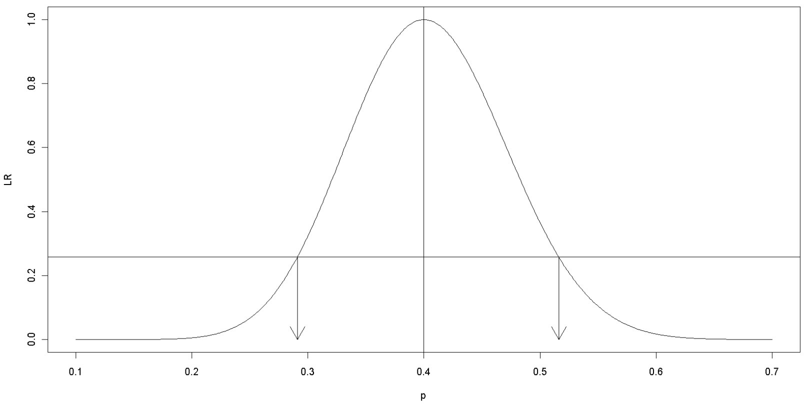


- Easy to see from the graph the most likely value of p is 0.4 ($L(0.4|x) = 9.77 \times 10^{-4}$).
- Absolute values of likelihood are tiny not easy to interpret
- Relative values of likelihood for different values of p are more interesting

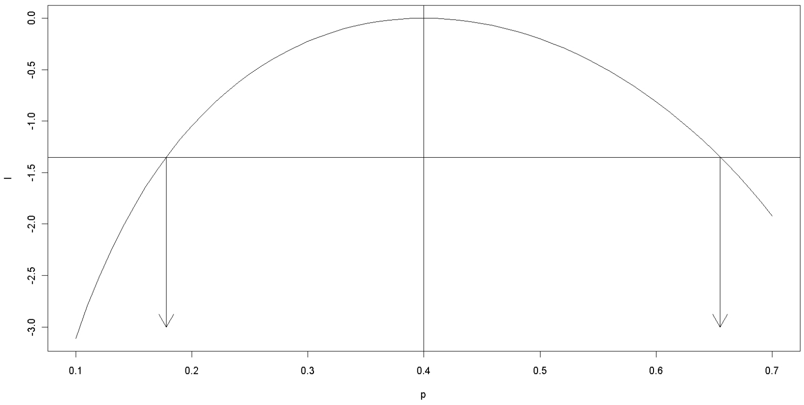
Plotting the Likelihood ratio:



- Measures how likely different values of p are relative to p=0.4.
- Can be used to define a group of values of p that are supported by the data, i.e. a group of values whose likelihood ratio is above a critical value.
- Using a critical value of 0.258 we find that the range of supported values is 0.178 to 0.655. This critical value corresponds to a 90% confidence level.
- If we had infact observed 20 deaths in a sample of 50 individuals, the most likely value of p would still be 0.4 but the supported range would be narrower: 0.291, 0.516.



Plotting the log-Likelihood ratio: The (log-)likelihood is invariant to alternative monotonic transformations of the parameter, so one often chooses a parameter scale on which the function is more symmetric.



Exercise: Tumble Mortality data: Write down the log likelihood function for the data on annealed glasses. Assume the shape parameter, μ, is known to be equal to 1.6. Plot the log likelihood function vs. possible values of the rate to determine the most plausible value of the rate for the observed data.