In the seventeenth century, French gamblers used to bet on the event that in 4 rolls of a die, at least one ace would turn up; an ace is •. In another game, they bet on the event that in 24 rolls of a pair of dice, at least one double-ace would turn up; a double-ace is a pair of dice which show • •.

The Chevalier de Méré, a French nobleman of the period, thought these two events were equally likely. He reasoned this way about the first game:

- In one roll of a die, I have 1/6 of a chance to get an ace.
- So in 4 rolls, I have $4 \times 1/6 = 2/3$ of a chance to get at least one ace.

His reasoning for the second game was similar:

- In one roll of a pair of dice, I have 1/36 of a chance to get a double-ace. So in 24 rolls, I must have $24 \times 1/36 = 2/3$ of a chance to get at least one double-ace.

By this argument, both chances were the same, namely 2/3. But experience showed the first event to be a bit more likely than the second. This contradiction became known as the *Paradox of the Chevalier de Méré.*

De Méré asked the philosopher Blaise Pascal about the problem, and Pascal solved it with the help of his friend, Pierre de Fermat. Fermat was a judge and a member of parliament, who is remembered today for the mathematical research he did after hours. Fermat saw that de Méré had used the addition rule for events that were not mutually exclusive. After all, it is possible to get an ace on both the first *and* second roll of a die. In fact, pushing de Méré's argument a little further, it shows the chance of getting an ace in 6 rolls of a die to be 6/6, or 100%. Something had to be wrong.

Now the question is how to calculate the chances correctly. Pascal and Fermat solved this problem, with a typically indirect piece of mathematical reasoning—the kind that always leaves non-mathematicians feeling a bit cheated. Of course, a direct attack like Galileo's (section 1) could easily bog down: with 4 rolls of a die, there are $6^4 = 1{,}296$ outcomes to worry about; with 24 rolls of a pair of dice, there are $36^{24} \approx 2.2 \times 10^{37}$ outcomes.

Unfortunately, the conversation between Pascal and Fermat is lost to history, but here is a reconstruction.[2]

In his novel *Bomber,* Len Deighton argues that a World War II pilot had a 2% chance of being shot down on each mission. So in 50 missions he is "mathematically certain" to be shot down: 50 x 2% = 100%. Is this a good argument?

Pascal Let's look at the first game first.

Fermat Bon. The chance of winning is hard to compute, so let's work out the chance of the opposite event: losing. Then

chance of winning = 100%—chance of losing.

Pascal D'accord. The gambler loses when none of the four rolls shows an ace. But how do you work out the chances?

Fermat It does look complicated. Let's start with one roll. What's the chance that the first roll doesn't show an ace?

Pascal It has to show something from 2 through 6, so the chance is 5/6.

Fermat C'est ça. Now, what's the chance that the first two rolls don't show aces?

Pascal We can use the multiplication rule. The chance that the first roll doesn't give an ace and the second doesn't give an ace equals $5/6 \times 5/6 = (5/6)^2$. After all, the rolls are independent, n'est-ce pas?

Fermat What about 3 rolls?

Pascal It looks like $5/6 \times 5/6 \times 5/6 = (5/6)^3$.

Fermat Oui. Now what about 4 rolls? Must be $(5/6)^4$.

Pascal Yes, and that's about 0.482, or 48.2%.

So there is a 48.2% chance of losing. Now

chance of winning = 100% - chance of losing
= 100% - 48.2% = 51.8%.

Fermat That settles the first game. The chance of winning is a little over 50%. Now what about the second?

Pascal. Well, in one roll of a pair of dice, there is 1 chance in 36 of getting a double-ace, and 35 chances in 36 of not getting a double-ace. By the multiplication rule, in 24 rolls of a pair of dice the chance of getting no double- aces must be $(35/36)^{24}$

Fermat That's about 50.9%. So we have the chance of losing. Now

chance of winning = 100%—chance of losing
= 100%—50.9% = 49.1%.

Pascal Yes, and that's a bit less than 50%. Voilà. That's why you win the second game a bit less frequently than the first. But you would have to roll a lot of dice to see the difference

THIS EXAMPLE ILLUSTRATES ONE GOOD STRATEGY FOR WORKING OUT CHANCES:

> If the chance of an event is hard to find, try to find the chance of the opposite event; then subtract from 100%. This is useful because the chance of the opposite event may be easier to compute