DISTRIBUTION OF WILCOXON SIGNED RANK STATISTIC: BY ENUMERATION

[JH] This statistic is calculated for a single sample (possibly a sample of intra-pair differences). One ranks the observations in absolute value, and then sums the ranks associated with positive observations to get $\sum+$ (the remainder, $\sum-$, can be calculated as n[n+1][2n+1] – $\sum+$ but it is not really needed for the analysis -- $\sum+$ and $\sum-$ are perfectly negatively correlated).

What distribution should this $\sum+$ have under H_0? Take a specific example, n=4. The ranks are 1, 2, 3, 4.

There are 2^4 = 16 configurations.

If all 4 observations are +, the signed ranks are +1, +2, +3, and +4, so that the sum of the positive ranks is 1+2+3+4=10, and there is only 1 such configuration.

If the signed ranks are –1, +2, +3, +4, then $\sum+$ = 9, and again, there is only 1 such configuration.

Likewise, there is only 1 way to get $\sum+$ = 8 (+1, –2, +3, +4). There are 2 ways to get $\sum+$ = 7 (–1, –2, +3, +4 and +1, +2, –3, +4), 2 ways to get $\sum+$ = 7, 5, 4, or 3. There is only 1 way to get $\sum+$ = 2, $\sum+$ = 1 or $\sum+$ = 0. These chances (out of 16) are all shown in the 11 diagonal cells corresponding to n=4.

By contrast, consider the distribution of the Sign Test statistic. Remember that the Sign Test just counts positive signs, without regard to magnitude. Thus, one can expect it to be cruder. One can see this 'coarseness' if one examines the possible values for the test statistic, which I call n+, the number of positive signs. Notice that of the 2^4 possibilities, there is a 1/16 chance that. n+ = 4, a 4/16 chance that n+ = 3, ..., a 1/16 chance that n+ = 0. Thus, the test statistic can take on only 5 possible values (4,3,2,1,0). By contrast, for the same sample size of n=4, the signed rank statistic can take on 11 possible values ($\sum+$ = 10 to $\sum+$ = 0). the "finer detail" in the signed rank statistic gives it more 'resolving power' than the sign test. The contrast in resolving power is even more evident if one calculates the 'grain' of the two distributions as a function of n:

Sample Size n	Number of distinct values for the... Sign Test Statistic	Signed Rank Statistic
4	5	11
5	6	16
6	7	22
7	8	29
8	9	37
9	10	46
10	11	56

Again, the null distribution is always symmetric. It takes somewhat longer than the Rank Sum Statistic to reach "Gaussian-ness".

It is possible to work out the exact null distribution for the Signed Rank Statistic for a sample of size n from the distribution for size n–1. For example, the entry of 6 corresponding the $\sum+$ of 19 for n=6 is obtained as the sum of the 5 immediately to its left and the 1 directly below it.

DISTRIBUTION OF SIGNED RANKS STATISTIC for n=1 to n=7

entries on diagonal represent frequency of each (∑-, ∑+) outcome {out of 2-to-power n}

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∑— \ ∑+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
28	1																												
27		1																											
26			1																										
25				2																									
24					2																								
23						3		*n = 7*																					
22							4				128	<-- number of possibilities = 2 to power of 7																	
21	1							5																					
20		1							5																				
19			1							6																			
18				2							7																		
17					2		*n = 6*					7																	
16						3			64				8																
15	1						4							8															
14		1						4							8														
13			1	*n =5*					4							8													
12				2			32			5							8												
11					2						5							7											
10	1					3						5							7										
9		1		*n = 4*			3						5							6									
8			1			16		3						4							5								
7				2					3						4							5							
6	1	*n = 3*			2					3						4							4						
5		1		8		2					3						3							3					
4			1				2					2						2							2				
3	1			2				2					2						2							2			
2		1			1				1					1						1							1		
1	1		1			1				1					1						1							1	
0		1		1			1				1					1						1							1