Appendix B

★ Some basic calculus

The *gradient* of the graph of y versus x measures the rate at which y is increasing (or decreasing) at any point on the graph. It is most easily defined for a straight line graph, such as the one in Fig. B.1. In this case the rate of increase or decrease is the same at any point on the graph, and is measured by the ratio of the *rise* to the *run*. For a straight line relationship in which y *decreases* with x the gradient is negative. Gradients have units equal to those of y/x. The central idea of calculus is that over a small run any curve is approximately a straight line and the gradient of the curve at any point in the run is approximately equal to the gradient of this line.

Differential calculus consists of a number of simple rules which are used to evaluate gradients of curves for which the y co-ordinate of any point on the curve is given by some function of the x co-ordinate. The most useful of these are shown in Table B.1. A further very important rule is that the gradient of a function constructed as the *sum* of two simpler functions is

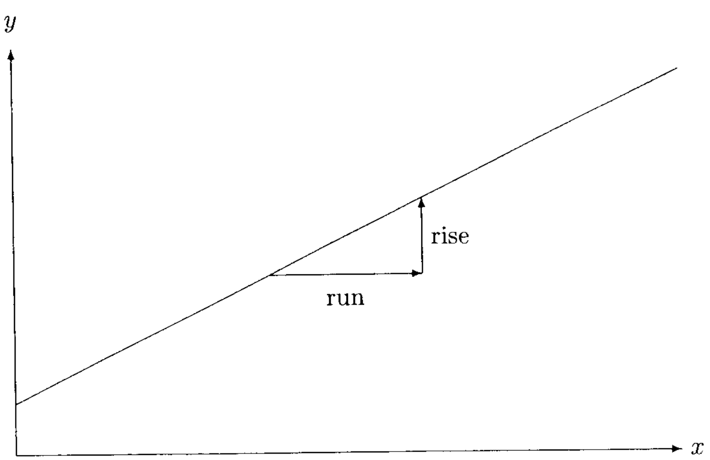

Fig. B.1. Gradient of a straight line graph

Table B.1. Gradients of some simple functions of x

Function	Gradient
c (constant)	0
x	1
$-x$	-1
cx	c
$(x)^2$	$2x$
$(x)^m$	$m(x)^{m-1}$
$\frac{1}{x} = (x)^{-1}$	$-(x)^{-2} = -\frac{1}{(x)^2}$
$\exp(x)$	$\exp(x)$
$\log(x)$	$\frac{1}{x}$
$(c+x)^2$	$2(c+x)$
$(c-x)^2$	$-2(c-x)$
$\log(c+x)$	$\frac{1}{c+x}$
$\log(c-x)$	$-\frac{1}{c-x}$

the sum of the gradients of the constituent functions so that, for example, the gradient of $x + \log(x)$ is $1 + 1/x$.

The use of these rules is now illustrated by finding the gradient of the log likelihood for a rate λ, based on D cases and Y person years. The log likelihood for λ is

$$D\log(\lambda) - \lambda Y.$$

From Table B.1 the gradient of $\log(\lambda)$ is $1/\lambda$ and the gradient of λ is 1. Hence the gradient of the log likelihood is

$$\frac{D}{\lambda} - Y.$$

The maximum value of the log likelihood occurs when the gradient is zero, that is, when $\lambda = D/Y$, so the most likely value of λ is D/Y.

The curvature of the log likelihood curve at the peak is important in determining the range of supported values. A highly curved peak corresponds to a narrow range. The curvature at a point on a curve is a measure of how fast the gradient is changing from one value of x to the next; if the gradient is changing quickly then the curvature is high, while if the gradient is changing slowly the curvature is low. For log likelihood curves the gradient changes from a positive quantity (on the left) to a negative quantity (on the right) so the gradient decreases as x increases and the curvature is negative.

The curvature of a curve, at a point, is defined to be the rate of change of the gradient of the curve at that point. The way that Table B.1 can be used to find curvature is now illustrated using the log likelihood for λ

again. The gradient of the log likelihood at any value of λ has been shown to be

$$\frac{D}{\lambda} - Y.$$

From Table B.1 the gradient of a constant is zero and the gradient of $1/\lambda$ is $-1/(\lambda)^2$, so the curvature of the log likelihood at any value of λ is

$$-\frac{D}{(\lambda)^2}.$$