Appendix A
Exponentials and logarithms

Raising 10 to different powers is a familiar operation. For example,

$$10^1 = 10, \ 10^2 = 100, \ 10^3 = 1000, \ \cdots$$

Mathematically this is regarded as a rule for getting from the power (1, 2, 3, etc.) to the value of 10 raised to that power (10, 100, 1000, etc.). The power is often referred to as the *exponent* and 10 raised to a power is called an *exponential* with base 10.

Raising 10 to a power can be extended to cover fractional powers using the convention that $10^{\frac{1}{2}}$ stands for the square root of 10, $10^{\frac{1}{3}}$ stands for the cube root of 10, and so on. The rule can also be extended to cover negative powers using the convention that 10^{-1} stands for $1/10 = 0.1$. Table A.1 shows the rule for obtaining 10^x from x for a variety of values of x.

Now suppose that we wish to go the other way and, starting with a value of 10^x, find the value of x. For example, starting with 1000 gives $x = 3$, while starting with 0.1 gives $x = -1$. Starting with any positive number y, the value of x which makes $10^x = y$ is called the *logarithm* of y with the base 10 and is written $\log_{10}(y)$. Taking logarithms with base 10 is the inverse operation to exponentiation with base 10. Thus $10^3 = 1000$ and $\log_{10}(1000) = 3$.

Table A.1. Rules for finding 10^x from x

x	$y = 10^x$
0	1
1	10
2	100
3	1000
−1	0.1
−2	0.01
−3	0.001
$\frac{1}{2}$	$\sqrt{10}$
$\frac{1}{3}$	$\sqrt[3]{10}$

Table A.2. Multiplication using logarithms

Number		Logarithm
7.2	⟶	0.8573
16.9	⟶	1.2279
121.7	⟵	2.0852

Logarithms were introduced as a computational device in the seventeenth century to avoid multiplication and division. Tables were prepared so that the logarithm of any number could be looked up. Similarly, tables of exponentials were prepared so that logarithms could be converted back to the original numbers. These tables of exponentials were called *antilogarithms.* The use of logarithms to multiply 7.2 by 16.9 is shown in Table A.2. Arrows from left to right refer to looking up logarithms while arrows from right to left refer to looking up antilogarithms (exponentiation). The result line follows from addition on the logarithmic (right-hand) side or multiplication on the exponential (left-hand) side. The widespread availability of cheap electronic calculators means that nobody now uses logarithms for multiplication or division. However, their mathematical property of converting multiplication to addition, embodied in

$$\log(7.2 \times 16.9) = \log(7.2) + \log(16.9)$$

is still very useful. Another useful property which follows from this is that

$$\log(7.2^2) = 2 \times \log(7.2)$$

$$\log(7.2^3) = 3 \times \log(7.2)$$

and so on.

Raising 2 to a power is called exponentiation with base 2. The inverse process produces logarithms to the base 2 and these are written $\log_2(y)$. Both exponentials and logarithms can be defined with respect to any base. Fig. A.1 shows plots of the exponential functions 10^x, 3^x, e^x, and 2^x, where the symbol e represents the number 2.71828183. The number e is chosen so that the tangent to the plot of e^x versus x drawn at $x = 0$ has a slope of exactly 1 (shown by the broken line). It follows that *when x is very small,*

$$e^x \approx 1 + x.$$

and, therefore,

$$\log_e(1 + x) \approx x.$$

Logarithms to the base e are referred to as *natural* logarithms, and it is the above property that makes them 'natural'. The natural logarithm

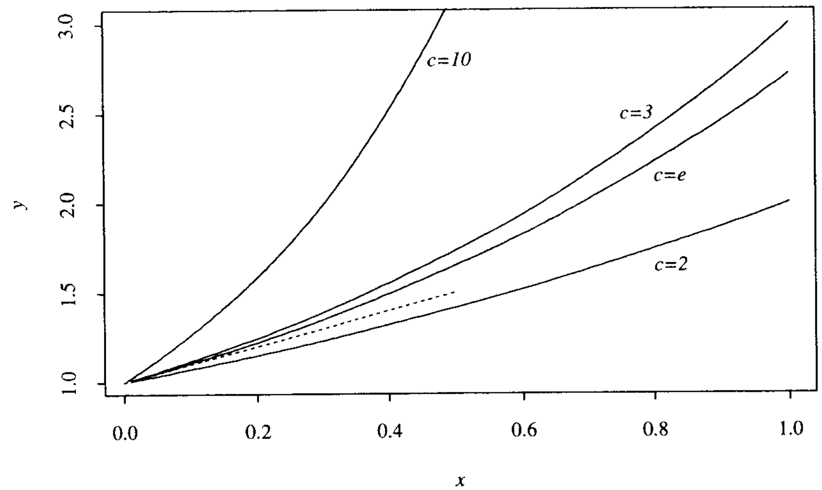

Fig. A.1. Plots of the function $y = c^x$

function is sometimes written as $\ln(y)$, but in this book we shall *always* use logarithms to the base e, and write them simply as $\log(y)$. We also write the exponential function with base e as $\exp(x)$. Note, however, that many electronic calculators assign an entirely different meaning to a key marked *exp*.

The logarithms of the same number, using different bases, are related by a simple constant multiplier. For example

$$\log_e(y) = \log_{10}(y) \times 2.3026$$

where $2.3026 = \log_e(10)$. Similarly

$$\log_2(y) = \log_{10}(y) \times 3.3219$$

where $3.3219 = \log_2(10)$.